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Paraproducts and Hankel operators of Schatten class via p-John–Nirenberg Theorem

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Abstract

We give an interpolation-free proof of the known fact that a dyadic paraproduct is of Schatten–von Neumann class S_p, if and only if its symbol is in the dyadic Besov space B_p^d. Our main tools are a product formula for paraproducts and a "p-John–Nirenberg-Theorem" due to Rochberg and Semmes.

We use the same technique to prove a corresponding result for dyadic paraproducts with operator symbols.

Using an averaging technique by Petermichl, we retrieve Peller's characterizations of scalar and vector Hankel operators of Schatten–von Neumann class S_p for $1<p<\infty$. We then employ vector techniques to characterise little Hankel operators of Schatten–von Neumann class, answering a question by Bonami and Peloso.

Furthermore, using a bilinear version of our product formula, we obtain characterizations for boundedness, compactness and Schatten class membership of products of dyadic paraproducts.

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1. Introduction and notation

In recent years, dyadic paraproducts have been successfully employed in the study of Hankel operators in various settings, see e.g. [Pet,PS]. In this paper, we want to look at Schatten class membership of scalar, vector and multivariable dyadic paraproducts and use these to study Schatten class membership of Hankel operators.

Boundedness, compactness and membership of Schatten classes of dyadic paraproducts have been characterised in terms of oscillatory properties of their symbols, see e.g. [ChPen,JPee,Pen,RS2] and the references within. Dyadic paraproducts on vector valued spaces (with matrix or more generally operator valued symbols) have also been studied, see e.g. [K,NPiTV]. It has not thus far been possible to characterise the boundedness of paraproducts with operator valued symbols in terms of oscillation properties of the symbol. These difficulties are closely connected with a breakdown of a form of the John–Nirenberg Theorem in the operator-valued setting (see [K,BlPo]). Here, we want to consider a "p-John–Nirenberg Theorem", which generalises easily to the operator setting.

The purpose of the paper is threefold. First, we show that a "p-John–Nirenberg Theorem" which can be found in [RS2] can be used to give a comparatively simple, interpolation-free proof of the characterisation of Schatten class paraproducts in terms of oscillatory properties of their symbols. Our approach is related to Rochberg and Semmes' method of nearly weakly orthonormal sequences—indeed, scalar dyadic paraproducts are in some sense the model case for nearly weakly orthonormal sequences, but technically simpler. Using an averaging technique from [Pet], it is possible to retrieve the known characterisation of Schatten class Hankel operators at least for $1<p<\infty$ (see [Pel1], for a second proof with different methods, see [CoR,R]). Our approach is again interpolation-free and has the advantage that one does not need any nontrivial properties of Besov spaces, for example the atomic decomposition of B_1^1.

Secondly, in contrast to the classical John–Nirenberg Theorem, the version of "p-John–Nirenberg Theorem" we require extends both to the operator-valued and the multivariable setting, and we thus obtain characterisations for the membership of Schatten classes for vector paraproducts and paraproducts in several variables. Again using averaging, we also obtain known results for vector Hankel operators (see [Pel2]) and new results on "little Hankel" operators, answering a question left open in [BoPelo1].

Finally, using a sesquilinear version of our method, we obtain necessary and sufficient conditions for boundedness, compactness and Schatten class membership of products of dyadic paraproducts. This part is motivated by the literature about products of Hankel operators, where characterisations of compactness of products of Hankel operators are known [Zh,ACS,V], but the corresponding questions about boundedness and Schatten class membership of products of Hankel operators are still open (see e.g. [TVZh,Pel3]).

The paper is organised as follows. In Section 2, we give a "p-John–Nirenberg" proof for the characterisation of dyadic paraproducts of Schatten class for $1 \leqslant p<\infty$. In Section 3, we characterise dyadic paraproducts of Schatten class with operator-valued symbols for $1 \leqslant p < \infty$. In Section 4, we give an interpolation-free proof of the characterisation of Hankel operators of Schatten class with operator symbols for $1 < p < \infty$. In Section 5, we use the vector results to characterise little Hankel operators of Schatten class on $H^2(\mathbb{C}^{+^n})$ and multivariable dyadic paraproducts of Schatten class. In Section 6, we characterise boundedness, compactness and Schatten class membership of products of paraproducts.

1.1. A dyadic decomposition

Let $\mathscr{D}$ denote the collection of all dyadic intervals on the real line, $\mathbb{R}$, so

$$\mathscr{D} := \{I = I_{n,k} := [2^{-n}k, 2^{-n}(k+1)) \colon n, k \in \mathbb{Z}\}.$$

Let $\mathscr{D}_n$ denote the collection of intervals in $\mathscr{D}$ of length 2^{-n}. For $I \in \mathscr{D}$, let $\tilde{I}$ denote the parent interval of I, let I_+ and I_- denote the left and right halves of I, respectively, $\mathscr{D}(I)$ the collection of dyadic intervals contained in I, $\mathscr{D}(I)'$ the collection of dyadic intervals contained properly in I, and $\mathscr{D}_n(I)$ the intersection of $\mathscr{D}_n$ and $\mathscr{D}(I)$. For $J \in \mathscr{D}'(I)$, we write $\operatorname{sign}(J, I) = 1$ for $J \in \mathscr{D}(I_{+)},$ $\operatorname{sign}(J, I) = -1$ for $J \in \mathscr{D}(I_-)$. We let h_I denote the Haar function corresponding to I, that is

$$h_I = \frac{1}{|I|^{1/2}} (\chi_{I_+} - \chi_{I_-}),$$

where χ_J denotes the characteristic function of an interval J. It is well known that $\{h_I \colon I \in \mathscr{D}\}$ forms an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. For further details on dyadic harmonic analysis, we refer to the survey article [Per].

Throughout the article, let C_p and K_p denote various constants, depending only on p.

1.2. Schatten classes

For a Hilbert space $\mathscr{H}$, let $\mathscr{L}(\mathscr{H})$ and $\mathscr{K}(\mathscr{H})$ denote the collections of bounded operators and compact operators on $\mathscr{H}$, respectively. Any operator $T \in \mathscr{K}(\mathscr{H})$ has a Schmidt decomposition, so there exist orthonormal bases $\{e_n\}$ and $\{\sigma_n\}$ of $\mathscr{H}$ and a sequence $\{\lambda_n\}$ with $\lambda_n \geqslant 0$ and $\lambda_n \to 0$ such that

$$Tf = \sum_{n=0}^{\infty} \lambda_n \langle f, e_n \rangle \sigma_n \tag{1}$$

for all $f \in \mathscr{H}$. For $0 < p < \infty$, a compact operator T with such a decomposition belongs to the Schatten–von Neumann p-class, $S_p(\mathscr{H})$, if and only if

$$||T||_{S_p} = \left(\sum_{n=0}^{\infty} \lambda_n^p \right)^{1/p} < \infty. \tag{2}$$

We shall frequently use the following elementary facts: For $0<p\leqslant 2$,

$$||T||_{S_p}^p = \inf\left\{\sum_{n\in\mathbb{N}} ||Te_n||^p\colon \{e_n\}_{n\in\mathbb{N}} \text{ orthonormal basis of } \mathscr{H}\right\} \tag{3}$$

for $2\leqslant p<\infty$,

$$||T||_{S_p}^p = \sup\left\{\sum_{n\in\mathbb{N}} ||Te_n||^p\colon \{e_n\}_{n\in\mathbb{N}} \text{ orthonormal basis of } \mathscr{H}\right\}, \tag{4}$$

see e.g. [G, p. 95].

2. Scalar dyadic paraproducts and Besov spaces

For a locally integrable function f on $\mathbb{R}$ and $I\in\mathscr{D}$, let $m_I f$ denote the mean value of f on I, i.e.

$$m_I f = \frac{1}{|I|}\int_I f(t)\,dt,$$

and f_I denote the Haar coefficient of f, i. e.

$$f_I = \langle f, h_I\rangle = \int_I f(t) h_I(t)\,dt.$$

For f locally integrable and $I\in\mathscr{D}$, we write $P_I f = \chi_I(f - m_I f) = \sum_{J\in\mathscr{D}(I)} h_J f_J$, and $P_I' f = \sum_{J\in\mathscr{D}'(I)} h_J f_J$. On $L^2(\mathbb{R})$, P_I and P_I' are the orthogonal projections on $\operatorname{span}\{h_J\colon J\in\mathscr{D}(I)\}$ and $\operatorname{span}\{h_J\colon J\in\mathscr{D}'(I)\}$, respectively.

We shall repeatedly use the following fact: for $I, J\in\mathscr{D}$,

$$m_J(h_I) = \pm\frac{1}{|I|^{1/2}}, \quad \text{when } J\in\mathscr{D}(I_\pm) \tag{5}$$

and zero otherwise. For a locally integrable function b, the densely defined dyadic paraproduct with symbol b, π_b is given by

$$\pi_b f = \sum_{I\in\mathscr{D}} m_I f b_I h_I.$$

It is easy to see that the adjoint of π_b on $L^2(\mathbb{R})$ is given by

$$\pi_b^* f = \sum_{I\in\mathscr{D}} \frac{\chi_I}{|I|} f_I \bar{b}_I. \tag{6}$$

We want to denote this adjoint operator by $\Lambda_{\bar{b}}$.

Necessary and sufficient conditions on the symbol b for π_b to be bounded on $L^2(\mathbb{R})$ or belong to a Schatten class have been obtained. We shall say a locally integrable function b belongs to the dyadic BMO space $\mathrm{BMO}^d(\mathbb{R})$ if

$$||b||_{\mathrm{BMO}^d} = \sup_{I\in\mathscr{D}} \frac{1}{|I|^{1/2}} ||P_I b|| < \infty.$$

Note that

$$\frac{1}{|I|^{1/2}}||P_I b|| = \left(\frac{1}{|I|}\int_I |b(t) - m_I b|^2\, dt\right)^{1/2} = \left(\frac{1}{|I|}\sum_{J\in\mathscr{D}(I)} |b_J|^2\right)^{1/2}.$$

We say that $b\in\mathrm{BMO}^d$ belongs to the dyadic VMO space $\mathrm{VMO}^d(\mathbb{R})$ if

$$\lim_{|I|\to 0} \frac{1}{|I|^{1/2}} ||P_I b|| = 0, \tag{7}$$

$$\lim_{|I|\to \infty} \frac{1}{|I|^{1/2}} ||P_I b|| = 0 \quad \text{and} \tag{8}$$

$$\lim_{|k|\to \infty} \frac{1}{|I_{n,k}|^{1/2}} ||P_{I_{n,k}} b|| = 0 \quad \text{for each } n\in\mathbb{Z}. \tag{9}$$

Here, the limits in (7) and (8) are meant to be *uniform* limits as $|I|\to 0$ or $|I|\to\infty$, respectively. Somewhat loosely, we will write

$$\lim_{I\to \infty} \frac{1}{|I|^{1/2}} ||P_I b|| = 0 \tag{10}$$

if conditions (7)–(9) above hold. We understand $I\to\infty$ as I converging to the point ∞ in the locally compact space $\mathscr{D}$ with the discrete topology.

For $0<p<\infty$, a locally integrable function b belongs to the dyadic Besov space $B_p^d(\mathbb{R})$ if

$$||b||_{B_p^d} = \left(\sum_{I\in\mathscr{D}} \left(\frac{|b_I|}{|I|^{1/2}}\right)^p\right)^{1/p} < \infty.$$

We then have

Theorem 2.1. (1) $b\in\mathrm{BMO}^d$ *if and only if* $\pi_b\in\mathscr{L}(L^2(\mathbb{R}))$;

(2) *For* $0<p<\infty$, $b\in B_p^d$ *if and only if* $\pi_b\in S_p(L^2(\mathbb{R}))$.

(3) $b \in \mathrm{VMO}^d$ *if and only if* $\pi_b \in \mathcal{K}(L^2(\mathbb{R}))$.

These results are well known, see e.g. [ChPen,Per] for the boundedness result and [ChPen,RS2] for the result concerning Schatten classes S_p, $1 \leqslant p < \infty$.

Before we give a "John–Nirenberg type" proof of Theorem 2.1(2), we need some more notation.

For a locally integrable function φ, let Q_φ be the so-called "dyadic sweep" or the square of the dyadic square function of φ, that is,

$$Q_\varphi(t) = \sum_{I \in \mathcal{D}} \frac{\chi_I(t)}{|I|} |\varphi_I|^2, \quad t \in \mathbb{R}. \tag{11}$$

We need the following elementary property of Q_φ.

Lemma 2.2. $P_I Q_\varphi = P_I Q_{P_I \varphi}$

Let D_φ be the operator on $L^2(\mathbb{R})$ which is diagonal in the Haar basis and defined by

$$D_\varphi h_I = h_I \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)'} |\varphi_J|^2 \quad (I \in \mathcal{D}).$$

The following identity relates the paraproducts π_φ and π_{Q_φ} (see [BlPo]).

Proposition 2.3.

$$\pi_\varphi^* \pi_\varphi = \pi_{Q_\varphi} + \pi_{Q_\varphi}^* + D_\varphi.$$

Proof. It suffices to show that $\langle \pi_\varphi^* \pi_\varphi h_I, h_J \rangle = \langle (\pi_{Q_\varphi} + \pi_{Q_\varphi}^* + D_\varphi) h_I, h_J \rangle$ for $I, J \in \mathcal{D}$. Note that π_φ is superdiagonal in the Haar basis in the sense that $\pi_\varphi h_I$ has nontrivial Haar coefficient only for $J \subsetneq I$, and π_φ is subdiagonal in the Haar basis in the sense that $\pi_\varphi h_I$ has nontrivial Haar coefficient only for $J \supsetneq I$.

Furthermore, $\operatorname{supp} \pi_\varphi h_I \subseteq I$ and $\operatorname{supp}(\pi_{Q_\varphi} + \pi_{Q_\varphi}^* + D_\varphi) h_I \subseteq I$ for all $I \in \mathcal{D}$, so we only have to consider the cases

(1) $I = J$:

$$\langle (\pi_{Q_\varphi} + \pi_{Q_\varphi}^* + D_\varphi) h_I, h_I \rangle = \langle D_\varphi h_I, h_I \rangle = \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)'} |\varphi_J|^2 \quad = \langle \pi_\varphi h_I, \pi_\varphi h_I \rangle$$

(2) $I \supsetneq J$:

$$\langle \pi_\varphi h_I, \pi_\varphi h_J \rangle = \sum_{K \in \mathcal{D}} |\varphi_K|^2 m_K h_I m_K h_J$$

$$= \frac{\operatorname{sign}(J,I)}{|I|^{1/2}|J|^{1/2}} \left(\sum_{K \in \mathscr{D}(J^+)} |\varphi_K|^2 - \sum_{K \in \mathscr{D}(J^-)} |\varphi_K|^2 \right)$$

$$= \frac{\operatorname{sign}(J,I)}{|I|^{1/2}} (Q_\varphi)_J = \langle \pi_{Q_\varphi} h_I, h_J \rangle = \langle (\pi_{Q_\varphi} + \pi^*_{Q_\varphi} + D_\varphi) h_I, h_J \rangle$$

(3) $I \subsetneq J$:

$$\langle \pi^*_{Q_\varphi} h_I, h_J \rangle = \langle h_I, \pi_{Q_\varphi} h_J \rangle = \langle h_I, \pi^*_\varphi \pi_\varphi h_J \rangle = \langle \pi^*_\varphi \pi_\varphi h_I, h_J \rangle$$

by (2). $\square$

We now need to temporarily introduce a further scale of function spaces. For $0 < p \leqslant \infty$ and $1 \leqslant q < \infty$, we say that $b \in L^2(\mathbb{R})$ belongs to the space $B^d_{p,q}$ if

$$||b||_{B^d_{p,q}} = \left(\sum_{I \in \mathscr{D}} \left(\frac{1}{|I|^{1/q}} ||P_I b||_q \right)^p \right)^{1/p} < \infty, \tag{12}$$

where $|| \ ||_q$ denotes the norm in $L^q(\mathbb{R})$. A continuous version of the following "p-John–Nirenberg Theorem" can be found in [RS2, Proposition 4.1].

Theorem 2.4. *Let $0 < p < \infty$. Then the spaces $B^d_{p,q}$, $1 \leqslant q < \infty$, all coincide with the dyadic Besov space B^d_p. The corresponding norms are equivalent.*

In the case $p = \infty$, all the spaces $B^d_{\infty,q}$ coincide with BMO^d, as known from the classical John–Nirenberg Theorem.

We shall present a proof of Theorem 2.4 here, which only uses the dyadic square function, a bootstrap argument and the following proposition from [RS1, Lemma 4.3], which covers the case $1 \leqslant q \leqslant 2$.

Proposition 2.5. *Let $0 < p < \infty$. There exists a constant α_p such that for each nonnegative sequence $(a_I)_{I \in \mathscr{D}}$ indexed by the dyadic intervals,*

$$\sum_{I \in \mathscr{D}} \left(\frac{1}{|I|} \sum_{J \in \mathscr{D}(I)} a_J \right)^p \leqslant \alpha_p \sum_{I \in \mathscr{D}} \left(\frac{a_I}{|I|} \right)^p.$$

Note that the reverse inequality trivially holds, with constant equal to 1. Note also that the $p = \infty$ version of the above statement fails, i.e. there exists no constant C such that

$$\sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} a_J \leqslant C \sup_{I \in \mathcal{D}} \frac{a_I}{|I|},$$

simply take $a_I = |I|$, for all $I \in \mathcal{D}$.

We include the proof of Proposition 2.5 for the convenience of the reader.

Proof of Proposition 2.5. We shall first suppose that $0 < p \leqslant 1$. Then, for all $I \in \mathcal{D}$, we have

$$\left(\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} a_J\right)^p \leqslant \frac{1}{|I|^p} \sum_{J \in \mathcal{D}(I)} a_J^p \quad \text{and hence}$$

$$\sum_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} a_J\right)^p \leqslant \sum_{I \in \mathcal{D}} \frac{1}{|I|^p} \sum_{J \in \mathcal{D}(I)} a_J^p = \sum_{J \in \mathcal{D}} \left(\sum_{k=0}^{\infty} \frac{1}{(2^k |J|)^p}\right) a_J^p,$$

as each dyadic interval J is contained in exactly one dyadic interval of size $2^k|J|$, for $k = 0, 1, \ldots$. Summing the infinite geometric series, we see that

$$\sum_{I \in \mathcal{D}} \left(\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} a_J\right)^p \leqslant \frac{2^p}{2^p - 1} \sum_{J \in \mathcal{D}} \left(\frac{a_J}{|J|}\right)^p$$

as required.

We now consider the case $1 < p < \infty$. We see that, for $I = I_{n,k} \in \mathcal{D}$,

$$\begin{aligned}
\left(\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} a_J\right)^p &= \left(2^n \sum_{m=n}^{\infty} \sum_{J \in \mathcal{D}_m(I)} a_J\right)^p \\
&= \left(\sum_{m=n}^{\infty} (m-n+1)^{-2}(m-n+1)^2 2^{n-m} \sum_{J \in \mathcal{D}_m(I)} 2^m a_J\right)^p \\
&\leqslant C_p \sum_{m=n}^{\infty} (m-n+1)^{2p-2} 2^{p(n-m)} \left(\sum_{J \in \mathcal{D}_m(I)} 2^m a_J\right)^p
\end{aligned}$$

for some constant C_p by Jensen's inequality, since $\sum_{m=n}^{\infty} (m-n+1)^{-2} = \pi^2/6$ for all m and $t \mapsto t^p$ is convex. Applying Hölder's inequality, where $1/p + 1/q = 1$, we see that

$$\left(\sum_{J\in\mathscr{D}_m(I)} 2^m a_J\right)^p \leqslant \sum_{J\in\mathscr{D}_m(I)} (2^m a_J)^p \left(\sum_{J\in\mathscr{D}_m(I)} 1^q\right)^{p/q}$$
$$= 2^{(m-n)(p-1)} \sum_{J\in\mathscr{D}_m(I)} (2^m a_J)^p$$

as $|\mathscr{D}_m(I_{n,k})| = 2^{m-n}$, when $m\geqslant n$. Consequently, we get

$$\sum_{I\in\mathscr{D}} \left(\frac{1}{|I|}\sum_{J\in\mathscr{D}(I)} a_J\right)^p \leqslant C_p \sum_{n,k\in\mathbb{Z}} \sum_{m=n}^{\infty} (m-n+1)^{2p-2} 2^{(n-m)} \sum_{J\in\mathscr{D}_m(I_{n,k})} (2^m a_J)^p$$
$$= C_p \sum_{m,j\in\mathbb{Z}} \sum_{n=-\infty}^{m} (m-n+1)^{2p-2} 2^{(n-m)} (2^m a_{I_{m,j}})^p,$$

changing the order of summation. But, for all $m\in\mathbb{Z}$,

$$\sum_{n=-\infty}^{m} (m-n+1)^{2p-2} 2^{(n-m)} = \sum_{l=1}^{\infty} l^{2p-2} 2^{-(l+1)} = K_p,$$

say. Therefore,

$$\sum_{I\in\mathscr{D}} \left(\frac{1}{|I|}\sum_{J\in\mathscr{D}(I)} a_J\right)^p \leqslant C_p K_p \sum_{J\in\mathscr{D}} \left(\frac{a_J}{|J|}\right)^p,$$

as required. □

Corollary 2.6. *Let b be a locally integrable function, and let $0<p<\infty$. Then $b\in B_p^d$ if and only if $b\in B_{p,q}^d$ for $1\leqslant q\leqslant 2$.*

Moreover, $||b||_{B_p^d}$ is equivalent to the expressions in (12).

Proof. Applying Hölder's inequality, it is easy to see that $B_{p,2}^d\subseteq B_{p,q}^d\subseteq B_{p,1}^d$ for $1\leqslant q\leqslant 2$, and it is also easy to see that $B_{p,1}^d\subseteq B_p^d$. All of these embeddings are bounded. So it only remains to prove that $B_p^d\subseteq B_{p,2}^d$, and that the embedding is bounded.

By Proposition 2.5,

$$||b||_{B^d_{p,2}} = \sum_{I\in\mathscr{D}} \left(\frac{1}{|I|}||P_I b||^2\right)^{p/2} = \sum_{I\in\mathscr{D}} \left(\frac{1}{|I|^{1/2}} \sum_{J\in\mathscr{D}(I)} |b_J|^2\right)^{p/2}$$
$$\leqslant \alpha_p \sum_{I\in\mathscr{D}} \left(\frac{|b_I|}{|I|^{1/2}}\right)^p = \alpha_p ||b||^p_{B^d_p}. \qquad \square$$

Before we can deal with the case $q>2$, we need

Proposition 2.7. *Let* $0<p<\infty$ *and* $1\leqslant q<\infty$. *Then*

$$||Q_b||_{B^d_{p,q}} \leqslant 2C^2_{2q}||b||^2_{B^d_{2p,2q}} \quad (b\in B^d_{2p,2q}),$$

where C_{2q} *is the norm of the dyadic square function on* $L^{2q}(\mathbb{R})$. *Conversely,*

$$||b||^2_{B^d_{2p,2q}} \leqslant c^2_{2q} l_p(||Q_b||_{B^d_{p,q}} + ||b||^2_{B^d_{2p,2}}),$$

where c_{2q} *is the lower bound of the dyadic square function on* $L^{2q}(\mathbb{R})$, *and* l_p *is a constant depending only on* p, $l_p = 1$ *for* $p\geqslant 1$.

Proof. Note that the projections $(P_I)_{I\in\mathscr{D}}$ are uniformly bounded on each $L^q(\mathbb{R})$, $1\leqslant q<\infty$, with norms bounded by 2, independent of q. Let $b\in B^d_{2p,2q}$. Then

$$||Q_b||_{B^d_{p,q}} = \left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{p/q}} ||P_I Q_b||^p_q\right)^{1/p} = \left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{p/q}} ||P_I Q_{P_I b}||^p_q\right)^{1/p}$$
$$\leqslant 2\left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{p/q}} ||Q_{P_I b}||^p_q\right)^{1/p}$$
$$\leqslant 2C^2_{2q}\left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{2p/2q}} ||P_I b||^{2p}_{2q}\right)^{1/p}$$
$$= 2\, C^2_{2q}||b||^2_{B^d_{2p,2q}},$$

where the first equality follows from Lemma 2.2.

Conversely,

$$\begin{aligned}
&\left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{2p/2q}} ||P_I b||_{2q}^{2p}\right)^{1/p} \\
&\leqslant c_{2q}^2 \left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{2p/2q}} ||Q_{P_I b}||_q^p\right)^{1/p} \\
&= c_{2q}^2 \left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{2p/2q}} ||P_I Q_{P_I b} + \chi_I m_I(Q_{P_I b})||_q^p\right)^{1/p} \\
&\leqslant c_{2q}^2 \left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{2p/2q}} \left(||P_I Q_b||_q + \frac{1}{|I|^{1-1/q}} ||Q_{P_I b}||_1\right)^p\right)^{1/p} \\
&= c_{2q}^2 \left(\sum_{I\in\mathscr{D}} \frac{1}{|I|^{2p/2q}} \left(||P_I Q_b||_q + \frac{1}{|I|^{1-1/q}} ||P_I b||_2^2\right)^p\right)^{1/p} \\
&\leqslant c_{2q}^2 l_p(||Q_b||_{B^d_{p,q}} + ||b||^2_{B^d_{2p,2}}). \qquad \square
\end{aligned}$$

Proof of Theorem 2.4. Let $2 < q < \infty$. Because of the trivial inclusion $B_{p,q_2} \subseteq B_{p,q_1}$ for $1 \leqslant q_1 \leqslant q_2 < \infty$, we can assume that $q = 2^n$, $n > 1$. We prove by induction over n that for all $n \in \mathbb{N}$ and all $0 < p < \infty$ there exists a constant $K_{p,n}$ such that

$$||b||_{B_{p,2^n}} \leqslant K_{p,n} ||b||_{B^d_p} \quad (b \in B^d_p).$$

By Corollary 2.6, this is true for $n = 1$. Suppose that the statement holds for some $n \in \mathbb{N}$. Then by Proposition 2.7, for each $b \in B^d_p$,

$$\begin{aligned}
||b||^2_{B^d_{p,2^{n+1}}} &\leqslant l_p c^2_{2^{n+1}}(||Q_b||_{B^d_{p/2,2^n}} + ||b||^2_{B^d_{p,2}}) \\
&\leqslant c^2_{2^{n+1}} l_p(K_{p/2,n} ||Q_b||_{B^d_{p/2,1}} + ||b||^2_{B^d_{p,2}}) \\
&\leqslant 2c^2_{2^{n+1}} l_p(K_{p/2,n} ||b||^2_{B^d_{p,2}} + ||b||^2_{B^d_{p,2}}) \\
&\leqslant ||b||^2_{B^d_p} 2\alpha_p^2 c^2_{2^n} l_p(K_{p/2,n} + 1).
\end{aligned}$$

The theorem follows now with an appropriate choice of $K_{p,n+1}$. $\square$

Corollary 2.8. *Let $0 < p < \infty$ and let $b \in B^d_p$. Then $Q_b \in B^d_{p/2}$, and there exists a constant $C_p > 0$ depending only on p such that $||Q_b||_{B^d_{p/2}} \leqslant C_p ||b||^2_{B^d_p}$.*

Proof. Corollary 2.6 (or Theorem 2.4) and Proposition 2.7. $\square$

Now we can give our "p-John–Nirenberg" proof of Theorem 2.1(2). We will give the full proof only for $p \geqslant 1$.

Proof of Theorem 2.1(2). Notice that

$$\sum_{I \in \mathscr{D}} ||\pi_b^* h_I||^p = \sum_{I \in \mathscr{D}} \left|\left| \bar{b}_I \frac{\chi_I}{|I|} \right|\right|^p = \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/2}} |b_I|^p = ||b||^p_{B_p^d}$$

for $0 < p < \infty$ by Proposition 2.5. Thus "$\Rightarrow$" follows immediately for $0 < p \leqslant 2$ from Eq. (3).

To prove "$\Rightarrow$" for $2 \leqslant p < \infty$, note first that for $0 < p < \infty$

$$\begin{aligned} ||D_b||^{p/2}_{S_{p/2}} = \sum_{I \in \mathscr{D}} |\langle \pi_b^* \pi_b h_I, h_I \rangle|^{p/2} &= \sum_{I \in \mathscr{D}} ||\pi_b h_I||^p \\ &= \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/2}} \left(\sum_{J \in \mathscr{D}'(I)} |b_J|^2 \right)^{p/2} \leqslant K_p ||b||^p_{B_p^d} \end{aligned}$$

by Proposition 2.5, and that therefore for $2 \leqslant p \leqslant 4$

$$\begin{aligned} ||\pi_b||^2_{S_p} = ||\pi_b^* \pi_b||_{S_{p/2}} &\leqslant 2||\pi_{Q_b}||_{S_{p/2}} + ||D_b||_{S_{p/2}} \\ &\leqslant 2||\pi_{Q_b}||_{S_{p/2}} + K_p ||b||^2_{B_p^d} \\ &\leqslant K_p'(||Q_b||_{B^d_{p/2}} + ||b||^2_{B_p^d}) \leqslant C_p ||b||^2_{B_p^d} \end{aligned}$$

by Corollary 2.8 and the first part of the proof. Inductively, we obtain the result for all p with $2 \leqslant p < \infty$. To obtain the reverse direction, we define a bounded operator R: $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ of norm 1 by $Rh_I = h_{\tilde{I}}$ for $I \in \mathscr{D}$, where $\tilde{I}$ denotes the parent interval of I. Recalling that

$$\sum_{n=1}^{\infty} |\langle Te_n, \sigma_n \rangle|^p \leqslant ||T||^p_{S_p} \tag{13}$$

for any orthonormal bases $\{e_n\}$, $\{\sigma_n\}$, $p \geqslant 1$ and $T \in S_p$ (this follows immediately from e.g. [G, p. 94]), we find that

$$\begin{aligned} ||\pi_b||^p_{S_p} \geqslant ||\pi_b R||^p_{S_p} \geqslant \sum_{I \in \mathscr{D}} |\langle Rh_I, \pi_b^* h_I \rangle|^p &= \sum_{I \in \mathscr{D}} \left| \left\langle h_{\tilde{I}}, \frac{\chi_I}{|I|} \bar{b}_I \right\rangle \right|^p \\ &= \frac{1}{2^{p/2}} \sum_{I \in \mathscr{D}} \left(\frac{|b_I|}{|I|^{1/2}} \right)^p = \frac{1}{2^{p/2}} ||b||^p_{B_p^d} \end{aligned}$$

for $1 \leqslant p < \infty$. $\square$

The implication "$\Leftarrow$" in Theorem 2.1(2) for $0<p<1$ is more difficult to deal with and was first shown by Peng [Pen]. We will return to this when looking at the operator case in the next chapter.

3. Operator valued Besov spaces and vector paraproducts of Schatten class

Dyadic paraproducts with matrix or operator symbols have been considered in e.g. [K,NPiTV,Pet]. We first introduce some notation for dyadic paraproducts acting on a vector valued Hilbert space, with operator valued symbols.

Let $\mathscr{H}$ denote separable Hilbert space and $L^2(\mathbb{R},\mathscr{H})$ the corresponding vector valued Hilbert space, so

$$L^2(\mathbb{R},\mathscr{H}) \coloneqq \{g\colon\ \mathbb{R}\to\mathscr{H}\colon\ \|g\|^2_{L^2(\mathbb{R},\mathscr{H})} = \int_{\mathbb{R}} \|g(t)\|^2_{\mathscr{H}}\, dt < \infty\}.$$

We may consider $L^2(\mathbb{R},\mathscr{H})$ as the Hilbert space tensor product $L^2(\mathbb{R})\otimes\mathscr{H}$ and, for $f\in L^2(\mathbb{R})$ and $x\in\mathscr{H}$, we let $f\otimes x$ denote the element of $L^2(\mathbb{R},\mathscr{H})$ defined for almost all $t\in\mathbb{R}$ by $f\otimes x(t) = f(t)x$.

Let B be a locally SOT-integrable operator valued function on $\mathbb{R}$, so $B(t)\in\mathscr{L}(\mathscr{H})$ for almost all $t\in\mathbb{R}$, and for $I\in\mathscr{D}$, we may formally define the operator $B_I\in\mathscr{L}(H)$ given by

$$\langle B_I x, y\rangle = \int_I h_I(t)\langle B(t)x, y\rangle\, dt, \quad x, y\in\mathscr{H}.$$

(For the definition of SOT integrability, see e.g. [Ni, Section 3.11].) We then define the (dyadic) paraproduct Π_B, acting on elementary tensors in $L^2(\mathbb{R},\mathscr{H})$ by

$$\Pi_B(f\otimes x) = \sum_{I\in\mathscr{D}} m_I f h_I\otimes B_I x, \quad f\in L^2(\mathbb{R}), x\in\mathscr{H} \tag{14}$$

and extending by linearity.

One would anticipate that the boundedness of such an operator would be characterised by an operator bounded mean oscillation criterion. However, it was shown in [NTV] that the naive generalisation of the scalar BMO condition to the operator case does not imply boundedness of the operator paraproduct.

We shall show, however, that Schatten class membership may be characterised by an operator Besov condition analogous to the scalar condition. These results can also be obtained using Rochberg and Semmes' method of nearly weakly orthonormal sequences from [RS1], although the vector case does not seem to appear in the literature.

We shall first derive an expression for Π_B^*.

Lemma 3.1. *If*

$$\Lambda_B(f \otimes x) = \sum_{I \in \mathscr{D}} f_I \frac{\chi_I}{|I|} \otimes B_I^* x, \quad f \in L^2(\mathbb{R}), x \in \mathscr{H}$$

extending by linearity, then $\Lambda_B = \Pi_B^*$.

Proof. This can easily be verified by means of elementary tensors. $\square$

We shall follow the same approach as in the scalar case, using dyadic square functions. For an operator valued function B, let Q_B be the "square of the dyadic square function" of B, that is

$$Q_B(t) = \sum_{I \in \mathscr{D}} B_I^* B_I \frac{\chi_I(t)}{|I|}, \quad t \in \mathbb{R}.$$

Let D_B be the operator on $L^2(\mathbb{R}, \mathscr{H})$ defined on elementary tensors by

$$D_B(f \otimes x) = \sum_{I \in \mathscr{D}} \frac{1}{|I|} \sum_{J \in \mathscr{D}(I)'} f_I h_I \otimes B_J^* B_J x.$$

The following identity relates the paraproducts Π_B and Π_{Q_B} (see [BlPo]).

Proposition 3.2.

$$\Pi_B^* \Pi_B = \Pi_{Q_B} + \Pi_{Q_B}^* + D_B.$$

Proof. It is sufficient to show that

$$\langle \Pi_B^* \Pi_B(h_I \otimes x), h_J \otimes y \rangle = \langle (\Pi_{Q_B} + \Pi_{Q_B}^* + D_B)(h_I \otimes x), h_J \otimes y \rangle \tag{15}$$

for all $I, J \in \mathscr{D}$ and $x, y \in \mathscr{H}$. This is shown exactly as in Proposition 2.3. $\square$

For $0 < p < \infty$, we shall say that an operator valued function B lies in the operator valued dyadic Besov space $\boldsymbol{B}_p^d$ if

$$||B||_{\boldsymbol{B}_p^d} = \left(\sum_{I \in \mathscr{D}} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}} \right)^p \right)^{1/p} < \infty.$$

We shall show that Schatten class operator valued dyadic paraproducts have symbols which belong to corresponding Besov spaces, thus generalising the scalar result.

We prove an operator analogue to Corollary 2.8. For $p \geqslant 2$, the Besov norms of B and Q_B are related in the same way as in the scalar case.

Lemma 3.3. *If* $2 \leqslant p < \infty$ *and* $B \in \boldsymbol{B}_p^d$ *then* $Q_B \in \boldsymbol{B}_{p/2}^d$, *with* $\|Q_B\|_{\boldsymbol{B}_{p/2}^d} \leqslant \alpha_p \|B\|_{\boldsymbol{B}_p^d}^2$, *for some universal constant* α_p.

Proof. Note that $\|\cdot\|_{S_{p/2}}$ is a norm. By the definition of Q_B and (5), we see that

$$(Q_B)_I = \frac{1}{|I|^{1/2}} \left(\sum_{J \in \mathscr{D}(I_+)} B_J^* B_J - \sum_{J \in \mathscr{D}(I_-)} B_J^* B_J \right),$$

and so $\|(Q_B)_I\|_{S_{p/2}} \leqslant \frac{1}{|I|^{1/2}} \sum_{J \in \mathscr{D}(I)'} \|B_J^* B_J\|_{S_{p/2}} = \frac{1}{|I|^{1/2}} \sum_{J \in \mathscr{D}(I)'} \|B_J\|_{S_p}^2,$

which gives $\|Q_B\|_{\boldsymbol{B}_{p/2}^d}^{p/2} \leqslant \sum_{I \in \mathscr{D}} \left(\frac{1}{|I|} \sum_{J \in \mathscr{D}(I)'} \|B_J\|_{S_p}^2 \right)^{p/2} \leqslant \alpha_p \|B\|_{\boldsymbol{B}_p^d}^p,$

by Proposition 2.5. □

The analogous statement for $p = \infty$ is false for infinite-dimensional $\mathscr{H}$ [BlPo]. For $I \in \mathscr{D}$, let U_I and V_I be the bounded operators given by

$$U_I\colon\ \mathscr{H} \to L^2(\mathbb{R}, \mathscr{H}), \quad U_I x = \chi_I \otimes x, \quad x \in \mathscr{H},$$

$$V_I\colon\ L^2(\mathbb{R}, \mathscr{H}) \to \mathscr{H}, \quad V_I F = \int_I F(t) h_I(t)\, dt, \quad F \in L^2(\mathbb{R}, \mathscr{H}).$$

Then $B_I = V_I \Pi_B U_I$. It follows that, for $0 < p < \infty$, if $\Pi_B \in S_p$ then $B_I \in S_p$.

Proposition 3.4. *If* $0 < p \leqslant 2$ *and* $B \in \boldsymbol{B}_p^d$ *then* $\Pi_B \in S_p$ *and* $\|\Pi_B\|_{S_p} \leqslant \|B\|_{\boldsymbol{B}_p^d}$;

Proof. Again, it will be more convenient to work with adjoints. Let $0 < p < \infty$ and $I \in \mathscr{D}$. Suppose (without loss of generality, by the discussion above) that B_I^* has a Schmidt decomposition,

$$B_I^* x = \sum_{n=0}^{\infty} \lambda_n^I \langle x, e_n^I \rangle \sigma_n^I, \quad x \in \mathscr{H},$$

where $\{e_n^I\}$ and $\{\sigma_n^I\}$ are orthonormal bases for $\mathscr{H}$. Therefore,

$$\|B_I^*\|_{S_p} = \|B_I\|_{S_p} = \left(\sum_{n=0}^{\infty} (\lambda_n^I)^p \right)^{1/p}.$$

It follows that $\{h_I \otimes e_n^I\colon I \in \mathscr{D},\ n = 0, 1, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R}, \mathscr{H})$. It is clear from Lemma 3.1 that $\Pi_B^*(h_I \otimes e_n^I) = \frac{\chi_I}{|I|} \otimes \lambda_n^I \sigma_n^I$. Consequently,

$$\sum_{n=0}^{\infty} ||\Pi_B^*(h_I \otimes e_n^I)||^p = \left(\frac{||B_I||_{S_p}}{|I|^{1/2}}\right)^p$$

for each $I \in \mathscr{D}$ and therefore

$$||\Pi_B||_{S_p}^p \leqslant \sum_{I \in \mathscr{D}} \sum_{n=0}^{\infty} ||\Pi_B^*(h_I \otimes e_n^I)||^p = ||B||_{\boldsymbol{B}_p^d}^p.$$

by (3). $\square$

The rest of this section will be concerned with showing that the statement in Proposition 3.4 extends to $p > 2$, and that also the reverse holds. We shall first use Proposition 3.4 and Lemma 3.3 to show that $B \in \boldsymbol{B}_p^d$ implies $\Pi_B \in S_p$, for $2 < p < \infty$.

Proposition 3.5. *If $2 \leqslant p < \infty$ and $B \in \boldsymbol{B}_p^d$ then $\Pi_B \in S_p$. Moreover, there exists a constant $C_p > 0$ depending only on p such that $||\Pi_B||_{S_p} \leqslant C_p ||B||_{\boldsymbol{B}_p^d}$.*

Proof. The proof runs along the lines of the proof of Theorem 2.1(2). We shall suppose that $2^n < p \leqslant 2^{n+1}$ for $n = 0, 1, \ldots$ and proceed by induction. The base case ($n = 0$) is covered by Proposition 3.4 so suppose that $n \geqslant 1$. We shall first consider the operator D_B. By definition, D_B has block diagonal form $D_B = (E_I)_{I \in \mathscr{D}}$ with respect to the Hilbert space decomposition $L^2(\mathbb{R}, \mathscr{H}) = \oplus_{I \in \mathscr{D}} \mathscr{H}$ given by $f \mapsto (f_I)_{I \in \mathscr{D}}$. Here, E_I is defined by $\langle E_I x, y \rangle = \langle \Pi_B^* \Pi_B h_I \otimes x, h_I \otimes y \rangle$ for $x, y \in \mathscr{H}$. That is, $E_I = \frac{1}{|I|} \sum_{J \in \mathscr{D}(I)'} B_J^* B_J$. Thus

$$\begin{aligned} ||D_B||_{S_{p/2}}^{p/2} = \sum_{I \in \mathscr{D}} ||E_I||_{S_{p/2}}^{p/2} &= \sum_{I \in \mathscr{D}} \left\| \frac{1}{|I|} \sum_{J \in \mathscr{D}(I)'} B_J^* B_J \right\|_{S_{p/2}}^{p/2} \\ &\leqslant \sum_{I \in \mathscr{D}} \left(\frac{1}{|I|^p} \sum_{J \in \mathscr{D}(I)'} ||B_J||_{S_p}^2 \right)^{p/2} \\ &\leqslant K_p \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/2}} ||B_I||_{S_p}^p = K_p ||B||_{\boldsymbol{B}_p^d} \end{aligned}$$

by Proposition 2.5, since $|| \cdot ||_{S_{p/2}}$ is a norm.

Also, by Lemma 3.3, $Q_B \in B_{p/2}$ and $2^{n-1} < p/2 \leqslant 2^n$. Hence, by the inductive hypothesis, $\Pi_{Q_B} \in S_{p/2}$ and $\|Q_B\|_{\boldsymbol{B}_p^d} \leqslant C_p \|B\|^2_{\boldsymbol{B}_p^d}$. Consequently, by Proposition 3.4, we have

$$\begin{aligned}\|\Pi_B\|^2_{S_p} = \|\Pi_B^* \Pi_B\|_{S_{p/2}} &\leqslant 2\|\Pi_{Q_B}\|_{S_{p/2}} + \|D_B\|_{S_{p/2}} \\ &\leqslant 2C_{p/2}\|Q_B\|_{\boldsymbol{B}^d_{p/2}} + \|D_B\|_{S_{p/2}} \leqslant C_p \|B\|^2_{\boldsymbol{B}_p^d} \qquad \square\end{aligned}$$

as required for an appropriate choice of C_p.

Finally, we must show that, for $0 < p < \infty$, $\Pi_B \in S_p$ implies that $B \in \boldsymbol{B}_p^d$. We will first deal with the case $1 \leqslant p < \infty$.

Proposition 3.6. *If $1 \leqslant p < \infty$ and $\Pi_B \in S_p$ then $B \in \boldsymbol{B}_p^d$, moreover $\|B\|_{\boldsymbol{B}_p^d} \leqslant C_p \|\Pi_B\|_{S_p}$ for some universal constant C_p.*

Proof. Let $1 \leqslant p < \infty$, and let T: $L^2(\mathbb{R}, \mathscr{H}) \to L^2(\mathbb{R}, \mathscr{H})$ be in the Schatten class S_p. The block diagonal $E = (E_I)_{I \in \mathscr{D}}$ of T, taken with respect to the Hilbert space decomposition $L^2(\mathbb{R}, \mathscr{H}) = \oplus_{I \in \mathscr{D}} \mathscr{H}$, $f \mapsto (f_I)_{I \in \mathscr{D}}$, is then given by

$$Eh_I \otimes e = h_I \otimes E_I e,$$

for $I \in \mathscr{D}$, where E_I: $\mathscr{H} \to \mathscr{H}$ is defined by $\langle E_I e, f \rangle = \langle Te \otimes h_I, f \otimes h_I \rangle$ for $I \in \mathscr{D}$, $e, f \in \mathscr{H}$. We will use the inequality

$$\|E\|^p_{S_p} = \sum_{I \in \mathscr{D}} \|E_I\|^p_{S_p} \leqslant \|T\|^p_{S_p}. \tag{16}$$

(see e.g. [G, p. 94]).

Similarly to the proof of Theorem 2.1(2), one defines a bounded linear operator R: $L^2(\mathbb{R}, \mathscr{H}) \to L^2(\mathbb{R}, \mathscr{H})$ of norm 1 by $Rh_I \otimes e = h_{\tilde{I}} \otimes e$ for $I \in \mathscr{D}$, $e \in \mathscr{H}$.

Suppose now that $\Pi_B \in S_p$. For each $I \in \mathscr{D}$, let B_I^* have the Schmidt decomposition

$$B_I^* x = \sum_{n=0}^{\infty} \lambda_n^I \langle x, e_n^I \rangle \sigma_n^I, \quad x \in \mathscr{H},$$

where $\{e_n^I\}$ and $\{\sigma_n^I\}$ are orthonormal bases for $\mathscr{H}$.

Applying (16) with $T = \Pi_B R$ and using (13), we obtain

$$
\begin{aligned}
||\Pi_B||_{S_p}^p &\geqslant ||\Pi_B R||_{S_p}^p \geqslant \sum_{I \in \mathscr{D}} ||E_I||_{S_p}^p \\
&\geqslant \sum_{I \in \mathscr{D}} \sum_{n=1}^{\infty} |\langle E_I \sigma_n^I, e_n^I \rangle|^p = \sum_{I \in \mathscr{D}} \sum_{n=1}^{\infty} |\langle \Pi_B R h_I \otimes \sigma_n^I, h_I \otimes e_n^I \rangle|^p \\
&= \sum_{I \in \mathscr{D}} \sum_{n=1}^{\infty} \left| \left\langle h_{\tilde{I}} \otimes \sigma_n^I, \frac{\chi_I}{|I|} B_I^* e_n^I \right\rangle \right|^p = \sum_{I \in \mathscr{D}} \frac{1}{2^{p/2}} \frac{1}{|I|^{p/2}} \sum_{n=1}^{\infty} |\lambda_n^I|^p \\
&= \frac{1}{2^{p/2}} \sum_{I \in \mathscr{D}} \left(\frac{||B_I||_{S_p}^p}{|I|^{1/2}} \right)^p = \frac{1}{2^{p/2}} ||B||_{\boldsymbol{B}_p^d}^p. \qquad \square
\end{aligned}
$$

Finally, we shall consider the case $0 < p < 1$. We shall generalise an argument found in [ChPen, Lemma 5.1]. Note that, for $0 < p < 1$, $|| \ ||_{S_p}$ is not a norm. However, if $T = R + S$, then

$$||T||_{S_p}^p \leqslant ||R||_{S_p}^p + ||S||_{S_p}^p, \tag{17}$$

see e.g. [Pen, p. 78]. We can obtain a partial reverse inequality, in the case that R and S have orthogonal ranges.

Lemma 3.7. *Let $0 < p < \infty$. If R and S are operators with orthogonal ranges and $T = R + S$ then*

$$||T||_{S_p}^p \geqslant \frac{1}{2} (||R||_{S_p}^p + ||S||_{S_p}^p).$$

Proof. It follows that $T^*T = R^*R + S^*S$, as $R^*S = S^*R = 0$. Therefore, $R^*R \leqslant T^*T$. By Douglas' Lemma (see e.g. [Yo, p. 143]), there exists a contraction Z such that $R = ZT$ and so $||R||_{S_p} \leqslant ||T||_{S_p}$. Similarly, $||S||_{S_p} \leqslant ||T||_{S_p}$, and the result follows. $\square$

For $m \in \mathbb{Z}$, we define the orthogonal projection Δ_m on $L^2(\mathbb{R}, \mathscr{H})$ by

$$\Delta_m(f \otimes x) = \sum_{I \in \mathscr{D}_m} \langle f, h_I \rangle h_I \otimes x,$$

defined here on elementary tensors for $f \in L^2(\mathbb{R})$ and $x \in \mathscr{H}$. We also define, for $m, n \in \mathbb{Z}$,

$$\Pi_B^{n,m} = \Delta_m \Pi_B \Delta_n.$$

Lemma 3.8. *Let $B \in \boldsymbol{B}_{\boldsymbol{p}}^{\boldsymbol{d}}$. If $m \leqslant n$ then $\Pi_B^{n,m} = 0$. If $m > n$ and $0 < p < 1$ then*

$$||\Pi_B||_{S_p}^p \leqslant 2^{(n-m)p/2} \sum_{I \in D_m} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}} \right)^p.$$

Proof. By definition, we see that, for $f \in L^2(\mathbb{R})$ and $x \in \mathscr{H}$,

$$\Pi_B^{n,m}(f \otimes x) = \sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m} \langle f, h_J \rangle m_I(h_J) h_I \otimes B_I x.$$

Therefore, by (5), if $m \leqslant n$ then $\Pi_B^{n,m} = 0$ and if $m > n$,

$$\Pi_B^{n,m}(f \otimes x) = \sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m(J_+)} \frac{\langle f, h_J \rangle}{|J|^{1/2}} h_I \otimes B_I x - \sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m(J_-)} \frac{\langle f, h_J \rangle}{|J|^{1/2}} h_I \otimes B_I x.$$

Let B_I have Schmidt decomposition

$$B_I x = \sum_{l=0}^{\infty} \lambda_l^I \langle x, e_l^I \rangle \sigma_l^I, \tag{18}$$

so we have

$$\begin{aligned} \Pi_B^{n,m}(f \otimes x) = &\sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m(J_+)} \sum_{l=0}^{\infty} \frac{\lambda_l^I}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_l^I \rangle h_I \otimes \sigma_l^I \\ &- \sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m(J_-)} \sum_{l=0}^{\infty} \frac{\lambda_l^I}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_l^I \rangle h_I \otimes \sigma_l^I. \end{aligned} \tag{19}$$

Thus, $\Pi_B^{n,m}$ has been expressed as the difference of two operators with given Schmidt decompositions, so, by (17),

$$\begin{aligned} ||\Pi_B^{n,m}||_{S_p}^p &\leqslant \sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m(J_+)} \sum_{l=0}^{\infty} \left(\frac{\lambda_l^I}{|J|^{1/2}} \right)^p + \sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m(J_-)} \sum_{l=0}^{\infty} \left(\frac{\lambda_l^I}{|J|^{1/2}} \right)^p, \\ &= \sum_{J \in \mathscr{D}_n} \sum_{I \in \mathscr{D}_m(J)} \left(\frac{||B_I||_{S_p}}{|J|^{1/2}} \right)^p \\ &= 2^{(n-m)p/2} \sum_{I \in D_m} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}} \right)^p. \qquad \square \end{aligned}$$

Let $B \in \boldsymbol{B}_p^d$ and N be a positive integer (to be determined later). For $k = 0, \dots N-1$, let

$$\Pi_{B,k} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Pi_B^{Nn+k,Nm+k+1} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m} \Pi_B^{Nn+k,Nm+k+1},$$

by Lemma 3.8. Let

$$\Pi_{B,k}^{(0)} = \sum_{n=-\infty}^{\infty} \Pi_B^{Nn+k,Nn+k+1}, \quad \Pi_{B,k}^{(1)} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m-1} \Pi_B^{Nn+k,Nm+k+1},$$

so $\Pi_{B,k} = \Pi_{B,k}^{(0)} + \Pi_{B,k}^{(1)}$.

Lemma 3.9. *For* $0 < p \leqslant 1$ *and* B, N *as above*,

$$\sum_{k=0}^{N-1} ||\Pi_{B,k}^{(0)}||_{S_p} \geqslant C_p ||B||_{\boldsymbol{B}_p^d}^p.$$

Proof. By (18) and (19), we see that,

$$\begin{aligned}
\Pi_B^{Nn+k,Nn+k+1}(f \otimes x) &= \sum_{J \in \mathscr{D}_{Nn+k}} \frac{\langle f, h_J \rangle}{|J|^{1/2}} (h_{J_+} \otimes B_{J_+} x - h_{J_-} \otimes B_{J_-} x) \\
&= \sum_{J \in \mathscr{D}_{Nn+k}} \sum_{l=0}^{\infty} \frac{\lambda_l^{J_+}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_l^{J_+} \rangle h_{J_+} \otimes \sigma_l^{J_+} \\
&\quad - \sum_{J \in \mathscr{D}_{Nn+k}} \sum_{l=0}^{\infty} \frac{\lambda_l^{J_-}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_l^{J_-} \rangle h_{J_-} \otimes \sigma_l^{J_-}
\end{aligned}$$

and so

$$\begin{aligned}
\Pi_{B,k}^{(0)}(f \otimes x) = &\sum_{n=-\infty}^{\infty} \sum_{J \in \mathscr{D}_{Nn+k}} \sum_{l=0}^{\infty} \frac{\lambda_l^{J_+}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_l^{J_+} \rangle h_{J_+} \otimes \sigma_l^{J_+} \\
&- \sum_{n=-\infty}^{\infty} \sum_{J \in \mathscr{D}_{Nn+k}} \sum_{l=0}^{\infty} \frac{\lambda_l^{J_-}}{|J|^{1/2}} \langle f \otimes x, h_J \otimes e_l^{J_-} \rangle h_{J_-} \otimes \sigma_l^{J_-}.
\end{aligned}$$

Since $\langle h_{J_+}, h_{I_-} \rangle = 0$ for all $I, J \in \mathscr{D}$, $\Pi_{B,k}^{(0)}$ has been expressed as the difference of two operators with orthogonal ranges and given Schmidt decompositions.

So, by Lemma 3.7,

$$\begin{aligned} \|\Pi_{B,k}^{(0)}\|_{S_p}^p &\geq \frac{1}{2}\left(\sum_{n=-\infty}^{\infty}\sum_{J\in\mathscr{D}_{Nn+k}}\sum_{l=0}^{\infty}\frac{(\lambda_l^{J_+})^p}{|J|^{p/2}} + \sum_{n=-\infty}^{\infty}\sum_{J\in\mathscr{D}_{Nn+k}}\sum_{l=0}^{\infty}\frac{(\lambda_l^{J_-})^p}{|J|^{p/2}}\right) \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{J\in\mathscr{D}_{Nn+k}}\frac{1}{|J|^{p/2}}(\|B_{J_+}\|_{S_p}^p + \|B_{J_-}\|_{S_p}^p) \\ &= \frac{1}{2^{p/2+1}}\sum_{n=-\infty}^{\infty}\sum_{I\in\mathscr{D}_{Nn+k+1}}\left(\frac{\|B_I\|_{S_p}^p}{|I|}\right)^{p/2}. \end{aligned}$$

Consequently,

$$\sum_{k=0}^{N-1}\|\Pi_{B,k}^{(0)}\|_{S_p}^p \geq \frac{1}{2^{p/2+1}}\sum_{I\in\mathscr{D}}\left(\frac{\|B_I\|_{S_p}^p}{|I|}\right)^{p/2} = \frac{1}{2^{p/2+1}}\|B\|_{\boldsymbol{B}_p^d}^p. \qquad \square$$

In seeking a converse to Proposition 3.4 for $0<p<1$, we shall first suppose that $B\in\boldsymbol{B}_p^d$; a simple density argument will then give the full result.

Proposition 3.10. *Let $0<p<1$. There exists a constant C_p such that if $B\in\boldsymbol{B}_p^d$ then $\|B\|_{\boldsymbol{B}_p^d}\leq C_p\|\Pi_B\|_{S_p}$.*

Proof. Note that

$$\Pi_{B,k} = \left(\sum_{n=-\infty}^{\infty}\Delta_{Nn+k}\right)\Pi_B\left(\sum_{m=-\infty}^{\infty}\Delta_{Nm+k+1}\right)$$

and so $\|\Pi_{B,k}\|_{S_p}\leq\|\Pi_B\|_{S_p}$, as $\sum_{n=-\infty}^{\infty}\Delta_{Nn+k}$ and $\sum_{m=-\infty}^{\infty}\Delta_{Nm+k+1}$ are norm 1 projections. Consequently,

$$\begin{aligned} N\|\Pi_B\|_{S_p}^p \geq \sum_{k=0}^{N-1}\|\Pi_{B,k}\|_{S_p}^p &\geq \sum_{k=0}^{N-1}(\|\Pi_{B,k}^{(0)}\|_{S_p}^p - \|\Pi_{B,k}^{(1)}\|_{S_p}^p) \\ &\geq C_p\|B\|_{\boldsymbol{B}_p^d}^p - \sum_{k=0}^{N-1}\|\Pi_{B,k}^{(1)}\|_{S_p}^p \qquad (20) \end{aligned}$$

by Lemma 3.9. However, for k $= 0, \dots, N-1$, we have by (17) and Lemma 3.8,

$$\begin{aligned}
||\Pi^{(1)}_{B,k}||^p_{S_p} &\leqslant \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m-1} ||\Pi_B^{Nn+k,Nm+k+1}||^p_{S_p} \\
&\leqslant \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{m-1} 2^{(Nn-Nm-1)p/2} \sum_{I\in\mathscr{D}_{Nm+k+1}} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}}\right)^p \\
&= \sum_{m=-\infty}^{\infty} 2^{-(Nm+1)p/2} \sum_{I\in\mathscr{D}_{Nm+k+1}} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}}\right)^p \sum_{n=-\infty}^{m-1} 2^{Nnp/2} \\
&= \sum_{m=-\infty}^{\infty} 2^{-(Nm+1)p/2} \sum_{I\in\mathscr{D}_{Nm+k+1}} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}}\right)^p \frac{2^{Nmp/2}}{2^{Np/2}-1} \\
&= \frac{2^{-p/2}}{2^{Np/2}-1} \sum_{m=-\infty}^{\infty} \sum_{I\in\mathscr{D}_{Nm+k+1}} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}}\right)^p.
\end{aligned}$$

Therefore,

$$\sum_{k=0}^{N-1} ||\Pi^{(1)}_{B,k}||^p_{S_p} \leqslant \frac{2^{-p/2}}{2^{Np/2}-1} \sum_{I\in\mathscr{D}} \left(\frac{||B_I||_{S_p}}{|I|^{1/2}}\right)^p = \frac{2^{-p/2}}{2^{Np/2}-1} ||B||^p_{\boldsymbol{B}_p^d}.$$

So, by (20), we see that, for all N,

$$||\Pi_B||^p_{S_p} \geqslant \frac{1}{N}\left(C_p - \frac{2^{-p/2}}{2^{Np/2}-1}\right)||B||^p_{\boldsymbol{B}_p^d}.$$

Choose N large enough so that $C_p - \frac{2^{-p/2}}{2^{Np/2}-1} > 0$ to obtain the required result. □

Corollary 3.11. *Let* $0<p\leqslant 1$. *If* $\Pi_B \in S_p$ *then* $B\in \boldsymbol{B}_p^d$. *Moreover, there exists a constant* C_p *such that* $||B||_{\boldsymbol{B}_p^d} \leqslant C_p ||\Pi_B||_{S_p}$.

Proof. For any positive integer N, let

$$\mathscr{D}^{(N)} := \{I_{n,k}\in\mathscr{D}\colon |n|\leqslant N, |k|\leqslant N\} \quad \text{and} \quad B^{(N)}(t) = \sum_{I\in\mathscr{D}^{(N)}} B_I h_I(t).$$

Then, $B^{(N)}\in \boldsymbol{B}_p^d$ and so $||B^{(N)}||_{\boldsymbol{B}_p^d} \leqslant C_p ||\Pi_{B^{(N)}}||_{S_p}$ by Proposition 3.10. But,

$$\Pi_{B^{(N)}}(f\otimes x) = \sum_{J\in\mathscr{D}^{(N)}} m_J f h_J \otimes B_J x = P^{(N)}\Pi_B(f\otimes x),$$

where $P^{(N)}$ is the orthogonal projection on $L^2(\mathbb{R}, \mathscr{H})$ defined by

$$\begin{aligned} P^{(N)}(h_J \otimes x) &= h_J \otimes x \quad \text{if } J \in \mathscr{D}^{(N)}, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{21}$$

Therefore,

$$\|B^{(N)}\|_{\boldsymbol{B_p^d}} \leqslant C_p \|P^{(N)} \Pi_B\|_{S_p} \leqslant C_p \|\Pi_B\|_{S_p}$$

for all N. But $\{\|B^{(N)}\|_{\boldsymbol{B_p^d}}\}$ is an increasing sequence and so we see that

$$\|B\|_{\boldsymbol{B_p^d}} = \lim_{N \to \infty} \|B^{(N)}\|_{\boldsymbol{B_p^d}} \leqslant C_p \|\Pi_B\|_{S_p}. \qquad \square$$

In summary, combining Propositions 3.4, 3.5, 3.11 and Corollary 3.6, we obtain our main result.

Theorem 3.12. *For $0 < p < \infty$, $\Pi_B \in S_p$ if and only if $B \in \boldsymbol{B_p^d}$. Moreover,*

$$C_p \|B\|_{\boldsymbol{B_p^d}} \leqslant \|\Pi_B\|_{S_p} \leqslant K_p \|B\|_{\boldsymbol{B_p^d}}.$$

4. From dyadic paraproducts to Hankel operators

Let $1 < p < \infty$, and let B: $\mathbb{R} \to \mathscr{L}(\mathscr{H})$ be locally integrable. We say that B is in the operator Besov space $\boldsymbol{B_p}(\mathbb{R})$, if

$$\|B\|_{\boldsymbol{B_p}}^p := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|B(x) - B(y)\|_{S_p}^p}{|x - y|^2} \, dx \, dy < \infty.$$

Theorem 4.1 (Peller [Pel2]). *Let $1 < p < \infty$, and let B: $\mathbb{R} \to \mathscr{L}(\mathscr{H})$ be antianalytic and locally integrable. Then the following are equivalent*:

(1) *The vector Hankel operator Γ_B: $L^2(\mathbb{R}, \mathscr{H}) \to L^2(\mathbb{R}, \mathscr{H})$ is in S_p.*
(2) $B \in \boldsymbol{B_p}(\mathbb{R})$.

We can use our results on vector paraproducts, along with the averaging procedure found in [Pet] to obtain an alternative proof of the sufficiency of this condition. One would expect $\boldsymbol{B_p}$ to be continuously included in $\boldsymbol{B_p^d}$, and we show this here.

Lemma 4.2. *Let $1<p<\infty$. Then there exists a constant $K_p>0$ such that if $B\in \boldsymbol{B_p}(\mathbb{R})$ then $B\in \boldsymbol{B_p^d}$ and $||B||_{\boldsymbol{B_p^d}}\leqslant K_p||B||_{\boldsymbol{B_p}}$.*

Proof. We use a method from [EJPenX]. It is easily shown that, for any $A\in S_p$ and $J\in\mathscr{D}$,

$$\frac{||B_J||_{S_p}}{|J|^{1/2}}\leqslant\frac{1}{|J|}\int_J ||B(x)-m_JB||_{S_p}\,dx\leqslant\frac{2}{|J|}\int_J ||B(x)-A||_{S_p}\,dx.$$

Letting $A=B(y)$ and then averaging for $y\in J$, we see that

$$\frac{||B_J||_{S_p}}{|J|^{1/2}}\leqslant\frac{2}{|J|^2}\int_J\int_J ||B(x)-B(y)||_{S_p}\,dx\,dy.$$

Supposing that $I\in\mathscr{D}_n$ for $n\in\mathbb{Z}$ and using Hölder's inequality, we see that

$$\begin{aligned}\sum_{J\in\mathscr{D}(I)}\frac{||B_J||_{S_p}^p}{|J|^{p/2}}&\leqslant 2^p\sum_{J\in D(I)}\frac{1}{|J|^2}\int_J\int_J ||B(x)-B(y)||_{S_p}^p\,dx\,dy\\&=2^p\sum_{m=n}^{\infty}\sum_{J\in\mathscr{D}_m(I)}\frac{1}{(2^{n-m}|I|)^2}\int_J\int_J ||B(x)-B(y)||_{S_p}^p\,dx\,dy\\&=\frac{2^p}{|I|^2}\int_{\mathbb{R}}\int_{\mathbb{R}} K_I(x,y)||B(x)-B(y)||_{S_p}^p\,dx\,dy,\end{aligned}$$

where

$$K_I(x,y)=\sum_{m=n}^{\infty}\sum_{J\in\mathscr{D}_m(I)}2^{2(m-n)}\chi_J(x)\chi_J(y).$$

Clearly, if either $x\notin I$ or $y\notin I$ then $K_I(x,y)=0$. Suppose that $x,y\in I$, with $x\neq y$ and let $J\in\mathscr{D}_m(I)$. Then $\chi_J(x)\chi_J(y)=0$ if $|x-y|>|J|=2^{n-m}|I|$. So,

$$K_I(x,y)\leqslant\sum_{m=n}^{n+[\log_2(|I|/|x-y|)]}2^{2(m-n)}\leqslant\frac{|I|^2}{3|x-y|^2}.$$

Therefore, for all $n\in\mathbb{Z}$,

$$\sum_{I\in\mathscr{D}_n}\sum_{J\in\mathscr{D}(I)}\frac{||B_J||_{S_p}^p}{|J|^{p/2}}\leqslant\frac{2^p}{3}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{||B(x)-B(y)||_{S_p}^p}{|x-y|^2}\,dx\,dy.$$

Letting $n \to -\infty$ we see that

$$||B||^p_{\boldsymbol{B}^d_p} = \sum_{J \in \mathscr{D}} \frac{||B_J||^p_{S_p}}{|J|^{p/2}} \leqslant \frac{2^p}{3} ||B||^p_{\boldsymbol{B}_p}. \qquad \square$$

For $\alpha \in \mathbb{R}$ and $r \in \mathbb{R}^+$, let $\mathscr{D}^{\alpha,r}$ denote the translated, dilated dyadic grid given by

$$\mathscr{D}^{\alpha,r} = \{[\alpha + r2^{-n}k, \alpha + r2^{-n}(k+1)) \colon n, k \in \mathbb{Z}\}.$$

For $J \in \mathscr{D}^{\alpha,r}$, let $h^{\alpha,r}_J$ denote the corresponding Haar function, normalised in $L^2(\mathbb{R})$. We define the dyadic shift $S^{\alpha,r}$ on $L^2(\mathbb{R})$ by

$$S^{\alpha,r}(f \otimes x) = \sum_{I \in \mathscr{D}^{\alpha,r}} \langle f, h^{\alpha,r}_I \rangle (h^{\alpha,r}_{I_+} \otimes x - h^{\alpha,r}_{I_-} \otimes x)$$

for an elementary tensor $f \otimes x \in L^2(\mathbb{R}, \mathscr{H})$. Note that $S^{\alpha,r}$ has norm $\sqrt{2}$. Let H: $L^2(\mathbb{R}, \mathscr{H}) \to L^2(\mathbb{R}, \mathscr{H})$ denote the vector Hilbert transform on $\mathbb{R}$, so

$$H(f \otimes x) = \left(p.v. \int_{\mathbb{R}} \frac{f(\cdot - y)}{y} \, dy \right) \otimes x.$$

Then, (see [Pet]), there exists a function $a \in L^\infty(\mathbb{R})$ and a constant $c_0 > 0$ such that the operator T: $L^2(\mathbb{R}, \mathscr{H}) \to L^2(\mathbb{R}, \mathscr{H})$ given by

$$T(f \otimes x) = c_0 H(f \otimes x) + (af) \otimes x \quad (f \in L^2(\mathbb{R}), x \in \mathscr{H})$$

is contained in the WOT-closed convex hull of the set $\{S^{\alpha,r} \colon \alpha \in \mathbb{R}, r \in \mathbb{R}^+\}$.

We begin by showing that for $1 < p < \infty$, there exists a constant $\tilde{C}_p > 0$ such that

$$||[S^{\alpha,r}, B]||_{S_p} \leqslant \tilde{C}_p ||B||_{\boldsymbol{B}_p} \quad (B \in \boldsymbol{B}_p). \tag{22}$$

For $\alpha \in \mathbb{R}$, $r \in \mathbb{R}^+$, let $\Pi^{\alpha,r}_B$ denote the vector paraproduct with respect to the dyadic grid $\mathscr{D}^{\alpha,r}$, given by

$$\Pi^{\alpha,r}_B f \otimes x = \sum_{I \in \mathscr{D}^{\alpha,r}} m_I f h^{\alpha,r}_I \otimes B_I x$$

and $\Lambda^{\alpha,r}_{B^*}$ its adjoint. Let $R^{\alpha,r}_B$ be the operator defined on elementary tensors by

$$R^{\alpha,r}_B f \otimes x = \sum_{I \in \mathscr{D}^{\alpha,r}} f_I h^{\alpha,r}_I \otimes m_I B x.$$

Here, $m_I f$, f_I, B_I and $m_I B$ denote Haar coefficients and averages with respect to the grid $\mathscr{D}^{\alpha,r}$. Then it is easily shown that $M_B = \Pi^{\alpha,r}_B + \Lambda^{\alpha,r}_B + R^{\alpha,r}_B$, see [Pet], and so

$$||S^{\alpha,r}M_B - M_B S^{\alpha,r}||_{S_p} \leqslant ||S^{\alpha,r}\Pi_B^{\alpha,r} - \Pi_B^{\alpha,r}S^{\alpha,r}||_{S_p} + ||S^{\alpha,r}\Lambda_{B^*}^{\alpha,r} - \Lambda_{B^*}^{\alpha,r}S^{\alpha,r}||_{S_p}$$
$$+ ||S^{\alpha,r}R_B^{\alpha,r} - R_B^{\alpha,r}S^{\alpha,r}||_{S_p}.$$

For F a function (scalar, vector or operator valued) defined on R, let

$$U^{\alpha,r}F(t) = r^{-1/2}F((t-\alpha)/r), \quad V^{\alpha,r}F(t) = F((t-\alpha)/r),$$

Then $U^{\alpha,r}$ is a unitary map on $L^2(\mathbb{R},\mathscr{H})$, $V^{\alpha,r}$ is an isometry on $\boldsymbol{B_p}$ and

$$U^{\alpha,r}\Pi_B^{\alpha,r}(U^{\alpha,r})^{-1} = \Pi_{V^{\alpha,r}B}. \tag{23}$$

If $B\in\boldsymbol{B_p}$, then

$$||S^{\alpha,r}\Pi_B^{\alpha,r} - \Pi_B^{\alpha,r}S^{\alpha,r}||_{S_p} \leqslant 2||S^{\alpha,r}||\,||\Pi_B^{\alpha,r}||_{S_p} = 2\sqrt{2}||\Pi_{V^{\alpha,r}B}||_{S_p}$$
$$\leqslant 2\sqrt{2}\,C_p||V^{\alpha,r}B||_{\boldsymbol{B_p}} = 2\sqrt{2}\,C_p||B||_{\boldsymbol{B_p}}$$

by (23), Theorem 3.12 and Lemma 4.2. It is similarly shown that $||S^{\alpha,r}\Lambda_B^{\alpha,r} - \Lambda_B^{\alpha,r}S^{\alpha,r}||_{S_p} \leqslant 2\sqrt{2}\,C_p||B||_{\boldsymbol{B_p}}$.

Finally, it may be seen that

$$(S^{\alpha,r}R_B^{\alpha,r} - R_B^{\alpha,r}S^{\alpha,r})(f\otimes x) = \sum_{I\in\mathscr{D}^{\alpha,r}} |I|^{-1/2}f_I(h_{I_+} - h_{I_-})\otimes B_I x.$$

If B_I has Schmidt decomposition $B_I = \sum \lambda_n^I\langle\cdot, e_n^I\rangle\sigma_n^I$, then

$$(S^{\alpha,r}R_B^{\alpha,r} - R_B^{\alpha,r}S^{\alpha,r})(f\otimes x) = \sum_{I\in\mathscr{D}^{\alpha,r}}\sum_{n=0}^{\infty}\frac{\sqrt{2}\,\lambda_n^I}{|I|^{1/2}}\langle f, h_I\rangle\left(\frac{h_{I_+} - h_{I_-}}{\sqrt{2}}\right)\otimes\langle x, e_n^I\rangle\sigma_n^I,$$

which is an expression in Schmidt form and so

$$||S^{\alpha,r}R_B^{\alpha,r} - R_B^{\alpha,r}S^{\alpha,r}||_{S_p}^p = \sum_{I\in\mathscr{D}^{\alpha,r}}\sum_{n=0}^{\infty}\frac{2^{p/2}(\lambda^I)^p}{|I|^{p/2}} = 2^{p/2}||B||_{\boldsymbol{B_p^{d^{\alpha,r}}}} \leqslant K_p||B||_{\boldsymbol{B_p}}^p$$

by Lemma 4.2, where $\boldsymbol{B_p^{d^{\alpha,r}}}$ is the dyadic Besov space defined with respect to $\mathscr{D}^{\alpha,r}$. This shows (22).

Let B: $\mathbb{R}\to\mathscr{L}(\mathscr{H})$, $B\in\boldsymbol{B_p}$, and suppose that B is locally bounded with respect to the operator norm on $\mathscr{L}(\mathscr{H})$. Let $(S_\gamma)_{\gamma\in\Gamma}$ be a net in $\operatorname{conv}\{S^{\alpha,r}\colon \alpha\in\mathbb{R}, r\in\mathbb{R}^+\}$ which converges to the operator T introduced above in the weak operator topology. It follows immediately from (22) that $||[S_\gamma, M_B]||_{S_p} \leqslant \tilde{C}_p||B||_{S_p}$. To proceed to the WOT-limit, we require the following elementary lemma.

Lemma 4.3. *Let $\mathscr{H}$ be a Hilbert space, let $1 \leqslant p < \infty$, and suppose that $(A_\gamma)_{\gamma \in \Gamma}$ is a bounded net of operators in $S_p \subseteq \mathscr{L}(\mathscr{H})$.*

Furthermore, suppose that there exists a dense subspace $\mathscr{A}$ of $\mathscr{H}$ and a sesquilinear form $\mathscr{A} \times \mathscr{A} \to \mathbb{C}$, $(x, y) \mapsto \langle Ax, y \rangle$, such that $\lim_{\gamma \in \Gamma} \langle A_\gamma x, y \rangle = \langle Ax, y \rangle$ for all $x, y \in \mathscr{A}$. Then A extends to a bounded linear operator on $\mathscr{H}$, $A \in S_p$, and $\|A\|_{S_p} \leqslant \sup_{\gamma \in \Gamma} \|A_\gamma\|_{S_p}$.

Proof. From

$$|\langle Ax, y \rangle| = \lim_{\gamma \in \Gamma} |\langle A_\gamma x, y \rangle| \leqslant \sup_{\gamma \in \Gamma} \|A_\gamma\| \|x\| \|y\| \leqslant \|x\| \|y\| \sup_{\gamma \in \Gamma} \|A_\gamma\|_{S_p}$$

for all $x, y \in \mathscr{A}$, it follows that A extends to a bounded sesquilinear form on $\mathscr{H} \times \mathscr{H}$, and therefore defines a bounded linear operator on $\mathscr{H}$, which we also denote by A. The estimate for the S_p norm of A is easily obtained from the identity

$$\|A\|_{S_p}^p = \sup \left\{ \sum_{n=1}^{N} |\langle Ae_n, \sigma_n \rangle|^p \colon N \in \mathbb{N}, \{e_n\}, \{\sigma_n\}_{n=1}^N \text{ orthonormal systems in } \mathscr{H} \right\}$$

see (13), and the density of $\mathscr{A}$ in $\mathscr{H}$. $\square$

We can now finish the proof of the direct implication in the theorem. Let $\mathscr{A} = \{f \in L^2(\mathbb{R}, \mathscr{H}), f \text{ has compact support}\}$. Since B is locally bounded, the commutator $[H, M_B]$ defines a sesquilinear form on $\mathscr{A} \times \mathscr{A}$, and one has

$$\lim_{\gamma \in \Gamma} \langle [S_\gamma, M_B]x, y \rangle = \langle [T, B]x, y \rangle = \frac{1}{c_0} \langle [H, B]x, y \rangle$$

by the WOT convergence of $(S_\gamma)_{\gamma \in \Gamma}$ to T. Thus by the previous lemma,

$$\|[H, M_B]\|_{S_p} \leqslant \frac{\tilde{C}_p}{c_0} \|B\|_{\boldsymbol{B_p}}. \tag{24}$$

It is not difficult to see that the locally bounded functions are dense in $\boldsymbol{B_p}$ for $p > 1$. For a given $B \in \boldsymbol{B_p}$, one can for example choose the sequence given by $B_n(x) = \tilde{B}(x + \frac{i}{n})$, where $\tilde{B}$ denotes the harmonic extension of B to the upper half plane. Then each B_n is locally bounded, and $(B_n)_{n \in \mathbb{N}}$ converges to B in $\boldsymbol{B_p}$ by the Dominated Convergence Theorem, subharmonicity and a vector version of Fatou's Thm, see e.g. [Ni, Section 3.11]. By density, we obtain (24) for all $B \in \boldsymbol{B_p}$. Using that $P^{\perp}_{H^2(\mathbb{R},\mathscr{H})}[H, M_B]P_{H^2(\mathbb{R},\mathscr{H})} = \frac{1}{2i}\Gamma_B$ for antianalytic B finishes the proof. $\square$

We note also that a version of Theorem 4.1 holds, for $0 < p \leqslant 1$, using appropriate definitions of the operator Besov spaces, for such p, see [Pel2].

5. Little Hankel operators and dyadic paraproducts in several variables

In this section, we want to use the vector valued results above to obtain a characterisation of Schatten class dyadic paraproducts in several variables and of Schatten class little Hankel operators on certain product domains.

As in the case of vector paraproducts, the method of nearly weakly orthonormal sequences from [RS2] provides an alternative route to obtain the characterisation of the symbols of Schatten class paraproducts, although this does appear not explicitly in the literature.

Let $n \in \mathbb{N}$. We write $\mathscr{R} = \mathscr{D}^n$ for the collection of dyadic rectangles in $\mathbb{R}^n$. For $R = I_1 \times \cdots \times I_n$, let $h_R(t_1, \ldots, t_n) = h_{I_1}(t_1) \cdots h_{I_n}(t_n)$. The collection $(h_R)_{R \in \mathscr{R}}$ is then the product Haar basis of $L^2(\mathbb{R}^n)$.

For a locally integrable function f on $\mathbb{R}^n$, we denote the Haar coefficient $\langle f, h_R \rangle$ by f_R, and the average $\frac{1}{|R|} \int_R f(t_1, \ldots, t_n)\, dt_1 \cdots t_n$ by $m_R f$.

Let $b \in L^2(\mathbb{R}^n)$. The densely defined linear mapping on $L^2(\mathbb{R}^n)$ given by

$$f \mapsto \sum_{R \in \mathscr{R}} h_R b_R m_R f \tag{25}$$

is the multivariable dyadic paraproduct with symbol b, denoted by π_b.

If we want to make clear that we take the paraproduct in n variables, we write $\pi_b^{(n)}$.

For $1 \leqslant i \leqslant n$, let P_i: $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ denote the Riesz projection in the ith variable and $P_i^{\perp}$ denote $I - P_i$. Then $P = P_1 \cdots P_n$ is the orthogonal projection from $L^2(\mathbb{R}^n)$ onto the Hardy space $H^2(\mathbb{R}^n)$. We identify functions in the Hardy space $H^2(\mathbb{C}^{+^n})$ on the n-fold product of the upper half planes with their boundary values in $H^2(\mathbb{R}^n)$ in the usual manner, and we write $H^2(\mathbb{R}^n) = H^2(\mathbb{C}^{+^n})$. Let $\bar{H}^2(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n)\colon \bar{f} \in H^2(\mathbb{R}^n)\}$.

The densely defined linear map on $H^2(\mathbb{R}^n) \to \bar{H}^2(\mathbb{R}^n)$ given by

$$f \mapsto P_1^{\perp} \cdots P_n^{\perp} bf \tag{26}$$

is the little Hankel operator with symbol b, denoted by γ_b. Again, we will write $\gamma_b^{(n)}$ if we want to emphasize that the Hankel operator is taken with respect to n variables.

The characterisations of bounded multivariable dyadic paraproducts and little Hankel operators in terms of their symbols are by no means a simple extension of the one-dimensional results.

For $n = 2$, boundedness of dyadic paraproducts was characterised in terms of an oscillation property of the symbol over all open sets in $\mathbb{R}^n$ by [Chang], and this gave rise to a characterisation of the dual of the Hardy space $H^1(\mathbb{C}^{+^n})$ in terms of oscillation properties [ChangFef]. Only recently, it was shown that also the boundedness of little Hankel operators on $H^2(\mathbb{C}^{+^2})$ can be characterised in terms of an oscillation property over open sets, in the course of the solution of the long-standing weak factorisation problem on $H^1(\mathbb{C}^{+^2})$ [FSa,FLa]. For $n \geqslant 3$, no such characterisation is known.

Little Hankel operators on the unit ball in $\mathbb{C}^n$, or more generally, on smoothly bounded strictly pseudoconvex sets, are much better understood [BoPelo2].

The main point of this section is to show that because of the good behaviour of Schatten class vector paraproducts and vector Hankel operators, multivariable paraproducts and little Hankel operators of Schatten class on certain product domains can be characterised quite easily in terms of their symbols.

In [BoPelo1], it is shown that for $b \in H^2(\mathbb{C}^{+^2})$, the little Hankel operator γ_b on $H^2(\mathbb{C}^{+^2})$ is of trace class, if and only if $\partial_1^2 \partial_2^2 b$ is integrable on $\mathbb{C}^{+^2}$, that is, b is in the Besov space $B_1(\mathbb{R}^2)$. This appears as a special case of a consideration of tube domains over symmetric cones. It is conjectured that these results extend at least to $1 < p \leqslant 2$ (see also [Ru] for an overview of known results and open questions of little Hankel operators on various types of domains in several dimensions).

We will give here a Besov space characterisation of the symbols for $1 < p < \infty$ for little Hankel operators, and for $0 < p < \infty$ for multivariable dyadic paraproducts.

Theorem 5.1. *Let $n \in \mathbb{N}$, $0 < p < \infty$, and let $b \in L^2(\mathbb{R}^n)$. Then $\pi_b \in S_p$ if and only if $\left(\sum_{R \in \mathscr{R}} \frac{1}{|R|^{p/2}} |b_R|^p\right)^{1/p} < \infty$, and the S_p norm of π_b is equivalent to this expression.*

Proof. We prove this statement by induction over n. For $n = 1$, this is just Theorem 2.1(2). Suppose that the statement is true for some $n \in \mathbb{N}$. Given $b \in L^2(\mathbb{R}^{n+1})$, we understand the multivariable paraproduct $\pi_b^{(n+1)}$ as a vector-valued paraproduct in one variable, defining $B(t) = \pi^{(n)}_{b(\cdot,\ldots,\cdot,t)}$ for $t \in \mathbb{R}$. We write b_I for the function on $\mathbb{R}^n$ given by $(t_1, \ldots, t_n) \to \int_I b(t_1, \ldots, t_n, t) h_I(t)\, dt$.

Then $\pi_{b_I}^{(n)} = B_I$, and it is easy to see that $\pi_b^{(n+1)}$: $L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}^{n+1})$ is unitarily equivalent to Π_B: $L^2(\mathbb{R}, L^2(\mathbb{R}^n)) \to L^2(\mathbb{R}, L^2(\mathbb{R}^n))$ via the natural unitary equivalence $L^2(\mathbb{R}^{n+1}) \to L^2(\mathbb{R}, L^2(\mathbb{R}^n))$.

Applying the induction hypothesis and Theorem 3.12, we obtain

$$\begin{aligned} ||\pi_b^{(n+1)}||_{S_p}^p = ||\Pi_B||_{S_p}^p \asymp \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/2}} ||B_I||_{S_p}^p &= \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/2}} ||\pi_{b_I}^{(n)}||_{S_p}^p \\ &\asymp \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/2}} \sum_{R' \in \mathscr{D}^n} \frac{1}{|R'|^{p/2}} |(b_I)_{R'}|^p \\ &= \sum_{R \in \mathscr{D}^{n+1}} \frac{1}{|R|^{p/2}} |b_R|^p. \qquad \square \end{aligned}$$

The same method applies for the characterisation of the symbols of little Hankel operators on $H^2(\mathbb{C}^{+^n})$ of Schatten class S_p, $1 < p < \infty$.

We need the following notation. For $i \in \{1, \dots, n\}$, $t \in \mathbb{R}$ and f: $\mathbb{R}^n \to \mathbb{C}$, let $\Delta_t^{(i)}$ be the finite difference operator in the ith coordinate given by

$$(\Delta_t^{(i)} f)(x) = f(x_1, \dots, x_i + t, \dots, x_n) - f(x_1, \dots, x_n) \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

For $1 < p < \infty$, we say that b: $\mathbb{R}^n \to \mathbb{C}$ is in $B_p(\mathbb{R}^n)$, if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|(\Delta_{t_1}^{(1)} \cdots \Delta_{t_n}^{(n)} b)(x)|^p}{\prod_{i=1}^n |t_i|^2} dx_1 \cdots dx_n \, dt_1 \cdots dt_n < \infty.$$

We denote the seminorm defined by the pth root of the expression above by $||b||_{B_p(\mathbb{R}^n)}$.

Applying the well-known equivalence of the "harmonic analysis" definition and the "complex analysis" definition of analytic Besov class functions coordinatewise (see e.g. [Ni, Theorem 8.7.2] for a version on the unit disc), one sees easily that for b analytic in $\mathbb{C}^{+^n}$, the expression on the left is equivalent to

$$\int_{\mathbb{C}^{+^n}} |\Im z_1|^{p-2} \dots |\Im z_n|^{p-2} |\partial_{z_1} \cdots \partial_{z_n} b(z)|^p \, dz_1 \cdots dz_n.$$

Theorem 5.2. *Let $1 < p < \infty$, and let $b \in \bar{H}^2(\mathbb{R}^n)$. Then the following are equivalent*:

(1) γ_b: $H^2(\mathbb{R}^n) \to \bar{H}^2(\mathbb{R}^n)$ *is in* S_p;
(2) $b \in B_p(\mathbb{R}^n)$,

and the $B_p(\mathbb{R}^n)$ norm is equivalent to the S_p norm.

Proof. For $n = 1$, this is just Peller's characterisation of Schatten class Hankel operators in the case $1 < p < \infty$. As before, we use induction over the dimension n. Suppose that the statement above holds for some $n \in \mathbb{N}$. Let $b \in \bar{H}^2(\mathbb{R}^{n+1})$. We define an operator valued function B: $\mathbb{R} \to \mathscr{L}(L^2(\mathbb{R}^n))$ by $B(t) = \gamma^{(n)}_{b(\cdot, \cdots, \cdot, t)}$. For each $t \in \mathbb{R}$, $b(\cdot, \cdots, \cdot, t)$ is an antianalytic function in n variables, and $b(\cdot, \cdots, \cdot, t) \in \bar{H}^2(\mathbb{R}^n)$ for almost every $t \in \mathbb{R}$. It is easy to verify that the vector Hankel operator Γ_B is unitarily equivalent to the little Hankel operator γ_b via the canonical unitary $L^2(\mathbb{R}, L^2(\mathbb{R}^n)) \to L^2(\mathbb{R}^{n+1})$. Therefore by Theorem 4.1,

$$\begin{aligned} ||\gamma_b^{(n+1)}||_{S_p}^p = ||\Gamma_B||_{S_p}^p &\asymp \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{||B(t) - B(s)||_{S_p}^p}{|t-s|^2} dt \, ds \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{||B(s+t) - B(s)||_{S_p}^p}{|t|^2} dt \, ds \end{aligned}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{||\gamma^{(n)}_{b(\cdot,\cdots,\cdot,s+t)} - \gamma^{(n)}_{b(\cdot,\cdots,\cdot,s)}||^p_{S_p}}{|t|^2} \, dt \, ds$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{||\gamma^{(n)}_{b(\cdot,\cdots,\cdot,s+t)-b(\cdot,\cdots,\cdot,s)}||^p_{S_p}}{|t|^2} \, dt \, ds$$

$$\asymp \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{||b(\cdot,\cdots,\cdot,s+t) - b(\cdot,\cdots,\cdot,s)||^p_{B_p(\mathbb{R}^n)}}{|t|^2} \, dt \, ds$$

$$= ||b||_{B_p(\mathbb{R}^{n+1})}. \qquad \square$$

The same method applies of course for little Hankel operators on domains of the form $D = \mathbb{C}^{+^n} \times \Omega \subseteq \mathbb{C}^{n+m}$ in the case where we have a Besov space type characterisation of Schatten class little Hankels on $H^2(\Omega)$, for example, if $\Omega \subseteq \mathbb{C}^m$ is a smoothly bounded convex domain of finite type (see [BoPelo2, Theorem 1.3]). For such domains, we can define the Hardy class $H^2(D) = H^2(\mathbb{C}^{+^n}) \otimes H^2(\Omega) \subseteq L^2(\mathbb{R}^n \times \partial\Omega)$ and, for $b \in \bar{H}^2(D)$, define the little Hankel operator γ_b on a dense subspace of $H^2(D)$.

Theorem 5.3. *Let $D = \mathbb{C}^{+^n} \times \Omega \subset \mathbb{C}^{n+m}$, where Ω is a smoothly bounded convex domain of finite type in $\mathbb{C}^m$. Let $b \in \bar{H}^2(D)$, and let $1 < p < \infty$.*

Then the following are equivalent.

(1)
$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{||\Delta^{(1)}_{t_1} \cdots \Delta^{(n)}_{t_n} b(x_1, \ldots, x_n, \cdot, \cdots, \cdot)||^p_{B_p(\Omega)}}{\prod_{i=1}^n |t_i|^2} \, dx_1 \cdots dx_n \, dt_1 \cdots dt_n < \infty;$$

(2) γ_b: $H^2(D) \to \bar{H}^2(D)$ *is in* S_p.

(For the definitions of $H^2(\Omega)$ and $B_p(\Omega)$, see [BoPelo2].) It would be interesting to see whether this method is also useful for domains of the form $\bigcup_{z \in \mathbb{C}^+} \{z\} \times \Omega_z \subset \mathbb{C}^{n+1}$, where Ω_z is a "sufficiently nice" domain in $\mathbb{C}^n$ for each $z \in \mathbb{C}^+$. The case of the light cone, which was studied by Bonami and Peloso [BoPelo1], would be an interesting candidate for this approach.

6. Products of dyadic paraproducts

Operator-theoretic properties of the product $\Gamma_f^* \Gamma_g$ of a Hankel operator and the adjoint of a Hankel operators have been studied for a long time, partly motivated by the identity $\Gamma_f^* \Gamma_g = [T_f, T_g)$, where $[T_f, T_g)$ denotes the semi-commutator $T_f T_g - T_{fg}$ of the Toeplitz operators T_f and T_g on $H^2(\mathbb{D})$, the Hardy space of the unit disc.

One example for this is the Axler–Chang–Sarason–Volberg Theorem, which characterises compact products of Hankel operators in terms of certain Douglas algebras ([V,ACS]).

The study of such products of Hankel operators is in general much more difficult than the study of single Hankel operators. There is still no full characterisation of boundedness and Schatten class membership in terms of oscillation properties of the symbols.

In [Zh], it was shown that the natural reproducing kernel condition

$$\lim_{|z|\to 1} ||\Gamma_g k_z||_2 ||\Gamma_f k_z||_2 = 0 \tag{27}$$

is equivalent to compactness of the product $\Gamma_f^* \Gamma_g$. Here, $\{k_z\}_{z\in\mathbb{D}}$ denote the normalised reproducing kernels on $H^2(\mathbb{D})$. However, it is an open question whether the reproducing kernel condition

$$\sup_{z\in\mathbb{D}} ||\Gamma_g k_z||_2 ||\Gamma_f k_z||_2 < \infty, \tag{28}$$

which can be understood as a "combined" oscillation condition and was shown to be necessary in [Zh], implies the boundedness of the product $\Gamma_g^* \Gamma_f$. Slightly stronger sufficient conditions have been found in [Zh,TVZh].

It is also open for which symbols $g, f \in L^2(\mathbb{T})$ the product of Hankel operators $\Gamma_g^* \Gamma_f$ is in the Schatten–von Neumann class S_p [Pel3], although partial results were found in [VI], and estimates for the singular values of such products have been obtained in [RS1, Section VII].

In this section, we will again consider dyadic paraproducts as a model case for Hankel operators and study operator-theoretic properties for products $\pi_f^* \pi_g$ of dyadic paraproducts. A sesquilinear version of the dyadic sweep from (11), given by

$$Q[f,g] = \sum_{I\in\mathscr{D}} \frac{\chi_I}{|I|} f_I \bar{g}_I \quad (f,g \in L^2(\mathbb{R})) \tag{29}$$

(see also [BlPo]), allows us to address this dyadic analogue in a very simple fashion.

As before, we collect some elementary properties of the sesquilinear map Q.

Lemma 6.1. (1) $||Q[f,g]||_1 \leqslant ||f||_2 ||g||_2$;
(2) $P_I Q[f,g] = P_I Q[P_I' f, P_I' g]$

We first need an analogue of Proposition 2.3. For $f, g \in L^2(\mathbb{R})$, let $D_{[f,g]}$ be defined on the Haar basis by

$$D_{[f,g]} h_I = \left(\frac{1}{|I|} \sum_{J\in\mathscr{D}'(I)} f_J \bar{g}_J \right) h_I.$$

Lemma 6.2.

$$\pi_g^*\pi_f = \pi_{Q[f,g]} + \Delta_{Q[f,g]} + D_{[f,g]}.$$

Proof. Exactly as in Proposition 2.3. □

Theorem 6.3. *Let $f, g \in L^2(\mathbb{R})$. Then the following are equivalent*:

(1) $\pi_g^*\pi_f$ *defines a bounded linear operator on* $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

(2) $Q[f,g] \in \mathrm{BMO}^d$, *and*

$$\sup_{I\in\mathscr{D}} \frac{1}{|I|}\left|\sum_{J\in\mathscr{D}'(I)} f_J\bar{g}_J\right| < \infty, \tag{30}$$

(3)

$$\sup_{I\in\mathscr{D}} \frac{1}{|I|} ||Q[P_I{}'f, P_I{}'g]||_1 < \infty,$$

(4)

$$\sup_{I\in\mathscr{D}} ||\pi_g^*\pi_f h_I|| < \infty.$$

Proof. (1) $\Rightarrow$ (4) obvious.

(4) $\Rightarrow$ (2): Remember from 6.2 that $\pi_{Q[f,g]}$ is the superdiagonal part of $\pi_g^*\pi_f$, $\Delta_{Q[f,g]}$ is the subdiagonal part, and $D_{[f,g]}$ is the diagonal part with respect to the Haar basis. The uniform boundedness of $||\pi_g^*\pi_f h_I||$ therefore implies uniform boundedness of $||D_{f,g}h_I|| = |\langle \pi_g^*\pi_f h_I, h_I\rangle|$ and thereby (30). Furthermore, note that $\frac{1}{|I|}\sum_{J\in\mathscr{D}'(I)} |(Q[f,g])_J|^2 = ||\pi_{Q[f,g]}h_I||^2 = ||P_I{}'\pi_g^*\pi_f h_I||^2 \leqslant ||\pi_g^*\pi_f h_I||^2$ for all $I \in \mathscr{D}$, and therefore $Q[f,g] \in \mathrm{BMO}^d$.

(2) $\Rightarrow$ (3): Note the identity $Q[P_I'f, P_I'g] = \chi_I m_I(Q[P_I{}'f, P_I{}'g]) + P_I(Q[f,g])$. Thus

$$\begin{aligned}\frac{1}{|I|} ||Q[P_I{}'f, P_I{}'g]||_1 &\leqslant |m_I(Q[P_I{}'f, P_I{}'g])| + \frac{1}{|I|} ||P_I(Q[P_I{}'f, P_I{}'g])||_1 \\ &= \frac{1}{|I|}\left|\sum_{J\in\mathscr{D}'(I)} f_J\bar{g}_J\right| + \frac{1}{|I|} ||P_I Q[f,g]||_1.\end{aligned} \tag{31}$$

(3) $\Rightarrow$ (2): By the uniform boundedness of the projections $(P_I)_{I\in\mathscr{D}}$ on $L^1(\mathbb{R})$, the uniform boundedness of the left-hand side in (31) implies the uniform boundedness of the right-hand side. Therefore $||Q[f,g]||_{\mathrm{BMO}^d} = \sup_{I\in\mathscr{D}} \frac{1}{|I|} ||P_I Q[f,g]||_1 < \infty$, and also (30) holds.

(2) $\Rightarrow$ (1): By Theorem 2.1, $\pi_{Q[f,g]}$ and $\Delta_{Q[f,g]}$ are bounded, and by (30), $D_{[f,g]}$ is bounded. Thus $\pi_g^*\pi_f = \pi_{Q[f,g]} + \Delta_{Q[f,g]} + D_{f,g}$ is bounded. $\square$

Condition (30) looks like a natural sesquilinear analogue to the BMO^d condition. However, a simple example shows that it is not sufficient for the boundedness of $\pi_g^*\pi_f$.

Remark 1. There exists functions $f, g \in L^2(\mathbb{R})$ such that (30) holds, but $\pi_g^*\pi_f$ does not define a bounded linear operator on $L^2(\mathbb{R})$.

Proof. Let f, g be defined by the following Haar coefficients. For $k \geqslant 0$, let $I_k = [0, 2^{-k}]$. Let $a > 0$, $1/2 < a^4 < 1$, and let $f_{I_k^+} = f_{I_k^-} = g_{I_k^+} = a^k$, $g_{I_k^-} = -a^k$ for each $k \geqslant 0$. Let all remaining Haar coefficients of f and g be 0. Then $g, f \in L^2(\mathbb{R})$ and $\sum_{J \in \mathscr{D}'(I)} f_J \bar{g}_J = 0$ for each $I \in \mathscr{D}$, but

$$\begin{aligned}\sum_{I \subseteq [0,1]} |(Q[f,g])_I|^2 &= \sum_{I \subseteq [0,1]} \frac{1}{|I|} \left| \sum_{J \in \mathscr{D}(I^+)} f_J \bar{g}_J - \sum_{J \in \mathscr{D}(I^-)} f_J \bar{g}_J \right|^2 \\ &= \sum_{k=0}^{\infty} \frac{1}{|I_k|} \left| \sum_{J \in \mathscr{D}(I_k^+)} f_J \bar{g}_J - \sum_{J \in \mathscr{D}(I_k^-)} f_J \bar{g}_J \right|^2 = \sum_{k=0}^{\infty} 2^k 4 a^{4k} \\ &= +\infty.\end{aligned}$$

The remark follows now by Theorem 6.3(2). $\square$

A further natural candidate for a "combined" BMO^d-condition is given by

$$\sup_{I \in \mathscr{D}} \frac{1}{|I|} \sum_{J \in \mathscr{D}'(I)} |f_J||g_J|. \tag{32}$$

It turns out that this condition leads even to a stronger property. For $\sigma \in \{-1, 1\}^{\mathscr{D}}$, let T_σ denote the dyadic martingale transform $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $\sum_{I \in \mathscr{D}} h_I f_I \mapsto \sum_{I \in \mathscr{D}} \sigma(I) h_I f_I$. Then we have the following result:

Theorem 6.4. *Let $f, g \in L^2(\mathbb{R})$. Then the following are equivalent*:

(1) *For each dyadic martingale transform T_σ, $\pi_g^* T_\sigma \pi_f$ defines a bounded linear operator on $L^2(\mathbb{R})$, and the operators $(\pi_g^* T_\sigma \pi_f)_{\sigma \in \{-1,1\}^{\mathscr{D}}}$ are uniformly bounded.*

(2)

$$\sup_{I \in \mathscr{D}} \frac{1}{|I|} \sum_{J \in \mathscr{D}'(I)} |f_J||g_J|.$$

(3)

$$\sup_{I\in\mathcal{D},\sigma\in\{-1,1\}^{\mathcal{D}}} ||\pi_g^* T_\sigma \pi_f h_I|| < \infty.$$

Proof. (1) $\Rightarrow$ (3) obvious.

(3) $\Rightarrow$ (2): Let $I\in\mathcal{D}$. Then for each $\sigma\in\{-1,1\}^{\mathcal{D}}$,

$$\begin{aligned}\langle \pi_g^* T_\sigma \pi_f h_I, h_I\rangle &= \frac{1}{|I|}\left\langle \sum_{J\in\mathcal{D}'(I)} \operatorname{sign}(J,I)\sigma(J) h_J f_J, \sum_{J\in\mathcal{D}'(I)} \operatorname{sign}(J,I) h_J g_J\right\rangle \\ &= \frac{1}{|I|}\sum_{J\in\mathcal{D}'(I)} \sigma(J)\bar{g}_J f_J.\end{aligned}$$

Choosing an appropriate sequence $(\sigma(J))_{J\in\mathcal{D}}\in\{-1,1\}^{\mathcal{D}}$, we obtain

$$\begin{aligned}\frac{1}{|I|}\sum_{J\in\mathcal{D}'(I)} |g_J||f_J| \leqslant \sqrt{2}\left|\sum_{J\in\mathcal{D}'(I)} \sigma(J)\bar{g}_J f_J\right| &\leqslant \frac{\sqrt{2}}{|I|}\sup_{\sigma\in\{-1,1\}^{\mathcal{D}}} |\langle \pi_g^* T_\sigma \pi_f h_I, h_I\rangle| \\ &\leqslant \frac{\sqrt{2}}{|I|}\sup_{\sigma\in\{-1,1\}^{\mathcal{D}}} ||\pi_g^* T_\sigma \pi_f h_I||.\end{aligned}$$

(2) $\Rightarrow$ (1): Observe that $\pi_g^* T_\sigma \pi_f = \pi_g^* \pi_{T_\sigma f}$. For all $I\in\mathcal{D}$ and all $\sigma\in\{-1,1\}^{\mathcal{D}}$,

$$\frac{1}{|I|}||Q[P_I{}'(T_\sigma f), P_I{}' g]||_1 = \frac{1}{|I|}\left\|\sum_{J\in\mathcal{D}'(I)} \frac{\chi_J}{|J|}\sigma(J)\bar{g}_J f_J\right\|_1 \leqslant \frac{1}{|I|}\sum_{J\in\mathcal{D}'(I)} |g_J|\,|f_J|.$$

(1) follows now from Theorem 6.3(3). $\square$

Unfortunately, when considering products of operators it is not as easy as in the situation in Section 4 to pass from results on paraproducts to results on Hankel operators via averaging. The following remark shows that products of paraproducts and products of Hankel operators behave quite differently.

Remark 2. A seemingly natural dyadic analogue to Zheng's necessary condition (28) is the following:

$$\sup_{I\in\mathcal{D}} ||\pi_f h_I||_2 ||\pi_g h_I||_2 < \infty. \tag{33}$$

This condition is easily seen to be sufficient for the uniform boundedness of all operator products $\pi_g^* T_\sigma \pi_f$, $\sigma\in\{-1,1\}^{\mathcal{D}}$, by Theorem 6.4(2).

However, whenever the sets $\{I\in\mathcal{D}\colon f_I\neq 0\}$ and $\{I\in\mathcal{D}\colon g_I\neq 0\}$ are disjoint, the product $\pi_g^* T_\sigma \pi_f$ is 0 for all $\sigma\in\{-1,1\}$. Thus one sees that (33) is not necessary.

Finally, we want to characterise Schatten class products of paraproducts. First, let us look at the compact case.

Theorem 6.5. *Let $f, g \in L^2(\mathbb{R})$. Then the following are equivalent*:

(1) $\pi_g^* \pi_f$ *defines a compact linear operator on* $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

(2) $Q[f, g] \in \mathrm{VMO}^d$, *and*

$$\lim_{I \to \infty} \frac{1}{|I|} \left| \sum_{J \subseteq I} f_I \bar{g}_I \right| = 0. \tag{34}$$

(3)

$$\lim_{I \to \infty} \frac{1}{|I|} ||Q[P_I{}'f, P_I{}'g]||_1 = 0.$$

(4)

$$\lim_{I \to \infty} ||\pi_g^* \pi_f h_I|| = 0.$$

Here, the limits in (2)–(4) are meant in the sense of (10), and convergence to 0 is meant to be *uniform*, as $|I| \to 0$ or $|I| \to \infty$, respectively.

Proof. (1) $\Rightarrow$ (4): For $N \in \mathbb{N}$, let $P^{(N)}$ denote the orthogonal projection defined in (21). $(P^{(N)})_{N \in \mathbb{N}}$ converges to the identity in the strong operator topology, so $\pi_g^* \pi_f P^{(N)} - \pi_g^* \pi_f$ converges to 0 in norm, and we obtain (4).

For the remainder of the proof, one shows (4) $\Rightarrow$ (2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (3) along the same lines as in the proof for Theorem 6.3, using Theorem 2.1(3) and Lemma 6.2. $\square$

Now we look at the Schatten classes S_p, $1 \leqslant p < \infty$. In this case, it turns out that if $\pi_g^* \pi_f \in S_p$ then also $\pi_g^* T_\sigma \pi_f \in S_p$ for all $\sigma \in \{-1, 1\}^{\mathscr{D}}$, with uniformly bounded S_p-norm. We get a natural combined dyadic Besov space condition for the symbols f and g.

Theorem 6.6. *Let $f, g \in L^2(\mathbb{R})$, $1 \leqslant p < \infty$. The following are equivalent*:

(i) $\pi_g^* T_\sigma \pi_f \in S_p$ *for each* $\sigma \in \{-1, 1\}^{\mathscr{D}}$, *and* $(\pi_g^* T_\sigma \pi_f)_{\sigma \in \{-1,1\}^{\mathscr{D}}}$ *is bounded in* S_p.

(ii) $\pi_g^* \pi_f \in S_p$.

(iii) $Q[f, g] \in B_p^d$, *and*

$$\sum_{I \in \mathscr{D}} \left| \frac{1}{|I|} \sum_{J \in \mathscr{D}'(I)} f_J \bar{g}_J \right|^p < \infty.$$

(iv)

$$\sum_{I\in\mathscr{D}} \frac{1}{|I|^p} |f_I g_I|^p < \infty .$$

(v) *For all* $1 \leqslant q < \infty$,

$$\sum_{I\in\mathscr{D}} \frac{1}{|I|^{p/q}} ||Q[P_I' f, P_I' g]||_q^p < \infty .$$

Proof. We will show
(ii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (iii) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii).
(ii) $\Rightarrow$ (iv): For $I \in \mathscr{D}$, let

$$\psi_I = \frac{1}{|I^{++}|^{1/2}} (\chi_{I^{+++}} - \chi_{I^{+--}})$$

and

$$\psi_I{}' = \frac{1}{|I^{++}|^{1/2}} (\chi_{I^{++-}} - \chi_{I^{--+}}).$$

The sequences $(\psi_I)_{I\in\mathscr{D}}$ and $(\psi_I{}')_{I\in\mathscr{D}}$ are not orthonormal, but it is easy to see that they are the images of the orthonormal Haar basis under bounded linear maps A, B. In the notation of [RS2], $(\psi_I)_{I\in\mathscr{D}}$ and $(\psi_I{}')_{I\in\mathscr{D}}$ are weakly orthonormal. Therefore,

$$\sum_{I\in\mathscr{D}} |\langle \pi_g^* \pi_f \psi_I, \psi_I{}' \rangle|^p = \sum_{I\in\mathscr{D}} |\langle B^* \pi_g^* \pi_f A h_I, h_I \rangle|^p \leqslant ||A||^p ||B||^p ||\pi_g^* \pi_f||_{S_p}^p .$$

Notice that

$$m_J \psi_I = \begin{cases} \dfrac{1}{2|I^{++}|^{1/2}} & \text{if } J = I^{++}, \\ -\dfrac{1}{2|I^{++}|^{1/2}} & \text{if } J = I^{+-}, \\ \dfrac{1}{|I^{++}|^{1/2}} & \text{if } J \subseteq I^{+++}, \\ -\dfrac{1}{|I^{++}|^{1/2}} & \text{if } J \subseteq I^{+--}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_J \psi_I{}' = \begin{cases} \dfrac{1}{4|I^{++}|^{1/2}} & \text{if } J = I^{+}, \\ -\dfrac{1}{4|I^{++}|^{1/2}} & \text{if } J = I^{-}, \\ \dfrac{1}{2|I^{++}|^{1/2}} & \text{if } J = I^{++}, \\ -\dfrac{1}{2|I^{++}|^{1/2}} & \text{if } J = I^{--}, \\ \dfrac{1}{|I^{++}|^{1/2}} & \text{if } J \subseteq I^{++-}, \\ -\dfrac{1}{|I^{++}|^{1/2}} & \text{if } J \subseteq I^{--+}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\langle m_J \psi_I, m_J \psi_I{}' \rangle$ equals $\frac{1}{4|I^{++}|}$ for $J = I^{++}$ and 0 otherwise, giving

$$\sum_{I \in \mathscr{D}} |\langle \pi_g^* \pi_f \psi_I, \psi_I{}' \rangle|^p = \frac{1}{4^p} \sum_{I \in \mathscr{D}} \frac{1}{|I^{++}|^p} |f_{I^{++}} g_{I^{++}}|^p.$$

Adjusting the definitions of ψ_I and $\psi_I{}'$, we obtain corresponding expressions for $\sum_{I \in \mathscr{D}} \frac{1}{|I^{+-}|^p} |f_{I^{+-}} g_{I^{+-}}|^p$, $\sum_{I \in \mathscr{D}} \frac{1}{|I^{-+}|^p} |f_{I^{-+}} g_{I^{-+}}|^p$, and $\sum_{I \in \mathscr{D}} \frac{1}{|I^{--}|^p} |f_{I^{--}} g_{I^{--}}|^p$. Thus (iv) holds.

(iv) $\Rightarrow$ (v): Let $\phi = \sum_{I \in \mathscr{D}} h_I |f_I g_I|^{1/2}$. Then

$$\begin{aligned} \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/q}} ||Q[P_I{}'f, P_I{}'g]||_q^p &\leqslant \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/q}} ||Q[P_I{}'\phi]||_q^p \\ &\leqslant C_{2q}^{2p} \sum_{I \in \mathscr{D}} \frac{1}{|I|^{p/q}} ||P_I{}'\phi||_{2q}^{2p} = C_{2q}^{2p} ||\phi||_{B^d_{2p,2q}}^{2p} \\ &\leqslant C_{2q}^{2p} K_{2p,2q} ||\phi||_{B^d_{2p}}^{2p} = C_{2q}^{2p} K_{2p,2q} \sum_{I \in \mathscr{D}} \frac{1}{|I|^p} |f_I g_I|^p. \end{aligned}$$

by Theorem 2.4, where C_{2q} denotes the norm of the dyadic square function on L^{2q}, and $K_{2p,2q}$ denotes the equivalence constant between the $B^d_{p,q}$ and the B^d_p norms from Theorem 2.4.

(v) $\Rightarrow$ (iii): Suppose that (v) holds for some q, $1 \leqslant q < \infty$. Then by Hölder's inequality, (v) holds in particular for $q = 1$. Note that the projections $(P_I)_{I \in \mathscr{D}}$ are uniformly bounded on $L^1(\mathbb{R})$. We obtain

$$\sum_{I\in\mathscr{D}} \frac{1}{|I|^p} ||P_I Q[f,g]||_1^p = \sum_{I\in\mathscr{D}} \frac{1}{|I|^p} ||P_I Q[P_I{}'f, P_I{}'g]||_1^p \leqslant C^p \sum_{I\in\mathscr{D}} \frac{1}{|I|^p} ||Q[P_I{}'f, P_I{}'g]||_1^p$$

and it follows from Theorem 2.4 that $Q[f,g]\in B_p^d$. Furthermore,

$$\sum_{I\in\mathscr{D}} \frac{1}{|I|} \left| \sum_{J\in\mathscr{D}'(I)} f_J \bar{g}_J \right|^p \leqslant \sum_{I\in\mathscr{D}} \frac{1}{|I|} ||Q[P_I{}'f, P_I{}'g]||_1^p.$$

Thus (iii) holds.

(iii) $\Rightarrow$ (ii): This follows directly from Lemma 6.2.

(iv) $\Rightarrow$ (i): Note again that $\pi_g^* T_\sigma \pi_f = \pi_g^* \pi_{T_\sigma f}$, and that condition (iv) is invariant under exchanging f with $T_\sigma f$.

Condition (i) now follows by applying the implication (iv) $\Rightarrow$ (ii) proved above to the symbols $T_\sigma f$ and g.

(i) $\Rightarrow$ (ii): This is immediate. $\square$

Using Theorem 3.12, we also obtain a vector version of this result:

Corollary 6.7. *Let $\mathscr{H}$ be a separable Hilbert space, let F, G: $\mathbb{R}\to\mathscr{L}(\mathscr{H})$ be weakly locally integrable, and let $1\leqslant p<\infty$. Then the following are equivalent:*

(i) $\Pi_G^* \Pi_F \in S_p$,

(ii)

$$\sum_{I\in\mathscr{D}} \frac{1}{|I|^p} ||G_I^* F_I||_{S_p}^p < \infty.$$

Proof. The proof $(ii) \Rightarrow (iv) \Rightarrow (i) \Rightarrow (ii)$ in Theorem 6.6 also works in the vector case. We omit the details here. $\square$

In [VI], it was shown that the condition

$$\sup_{k\in\mathbb{N}} \sum_{I\in\mathscr{D}_k} \frac{1}{|I|^p} |f_I g_I|^p < \infty \tag{35}$$

is necessary for the product of Hankel operators $\Gamma_g^* \Gamma_f$ to be in S_p.

We have seen above that the stronger condition Theorem 6.6 (iv) holds whenever $\pi_g^* \pi_f$ is in S_p. It would be interesting to know whether Theorem 6.6 (iv) holds (at least in some average sense) whenever $\Gamma_g^* \Gamma_f$ is in S_p. Conversely, it would be of great interest to know whether a translation and dilation invariant version of condition 6.6 (iv) implies that $\pi_g^* \pi_f$ is in S_p. We finish by stating this as a conjecture.

As before, denote by $\mathscr{D}^{\alpha,r}$, where $\alpha \in \mathbb{R}$ and $r \in \mathbb{R}^+$, the dyadic grid obtained by dilating the standard dyadic grid $\mathscr{D}$ by r and then translating it by α.

Conjecture 6.8. *Let* $1 \leqslant p < \infty$, *and let* $f, g \in \bar{H}^2(\mathbb{R})$. *Suppose that*

$$\sup_{\alpha \in \mathbb{R}, r \in \mathbb{R}^+} \sum_{I \in \mathscr{D}^{\alpha,r}} \frac{1}{|I|^p} |f_I g_I|^p < \infty.$$

Then $\Gamma_g^* \Gamma_f \in S_p$.

Remark: After this paper had been refereed, V.V. Peller pointed out to us that a characterisation of little Hankel operators of Schatten class on the bidisk can be found in the article [LiPen] (in Chinese).

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